UNIT 5 LIMITS AND CONTINUITY

Structure **Page Nos.**

5.1 INTRODUCTION

The passage to calculus and analysis is through the gateway of 'limits'. Unfortunately, this concept is a common stumbling point for most fresh calculus students. Therefore, they somehow manage to cope with calculus without really trying to understand this concept, and its allied concept of 'continuity'. In particular, students usually confuse the limit with the value of a function. In this unit, in Sec.5.2, we give some suggestions and strategies to help students get over such misunderstandings. We also suggest some teaching strategies /classroom activities that may help the students understand the concept in a better way.

In the next section, Sec.5.3, we discuss continuity. The focus of the discussion is teaching strategies for overcoming students' confusions and errors regarding this concept and its complement, 'discontinuity'.

Apart from this unit, we have developed a related CD, 'Limits — A Glance Through History', which you can see at the Extended Contact Programme, or when it is broadcast on DD-1.

Objectives

After studying this unit, you should be able to develop the ability of your learners to

- explain why the limit of a function at a point in its domain exists, and to find it if it exists;
- explain why a function is continuous (or otherwise) at a given point.

5.2 INTRODUCING LIMITS

When we use the word 'limit' in ordinary conversation, we mean 'boundary' or 'extreme'. Is this the meaning in the mathematical context? Can we help students to relate the two meanings so that they can develop an intuitive understanding of this so-called 'difficult' concept? Giving our learners real-life examples from the world around us, like the following ones, may help in this matter.

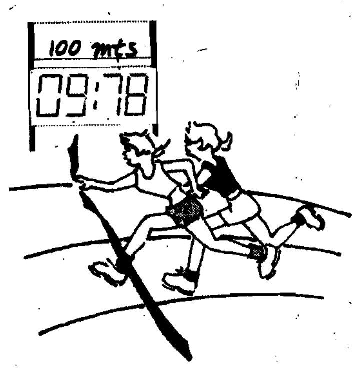

Fig. 1

You could ask your students to consider the time duration for running a 100 metre race. There have been many world records created and broken in this. It has been run in a record time of 9.79 seconds. Maybe the record is bettered in future — 9.78 secs., 9.77 secs., etc. But, it is likely that no one will ever be able to finish the race in, say, 3 seconds. So, there is a duration that no one will be able to better, but records will get closer and closer to it. Let us assume that this duration is 9.7 seconds. This would be the minimum possible duration for completing the race. In a sense, 9.7

seconds is a boundary which will not be crossed, but is nearly reached. This would be the limit of the duration for running a 100 metre race.

As another example, you could ask your students to consider the speed at which a car moves. The greatest possible speed of the car is, say, 160 km/hr. As the car goes faster and faster, you see the needle of the speedometer moving from 100 to 120 km/hr., and on to 130 km/hr., 140,..., 159 km/hr., and possibly 160 km/hr. But it can't go further. So the upper limit of the speed of the car is 160 km/hr. What is the lower limit? As a car slows, its speed moves to this level gradually, and finally attains this limit, namely, 0 km/hr., when the car stops.

You can think of many other examples. Here are some related exercises.

E1) Write down 3 other examples of limits related to the day-to-day life experiences of your students.

E2) An equilateral triangle is inscribed in a circle of radius 1, and a circle is then inscribed in the triangle. This process continues, and we get the pattern

Circle $\supseteq$ Triangle $\supseteq$ Circle $\supseteq$ Triangle $\supseteq$ Circle $\supseteq$ ...

Ask your students to find if there is a pattern that emerges in the radii of the circles. What is the limiting value as the radii become smaller and smaller?

Understanding a concept is helped greatly if the students see why the concept is needed. You could give them an informal preview about how 'limits' are a basic concept for developing calculus, and for understanding the behaviour of functions in general. For example, they could be told why some series are meaningless because the sum doesn't converge. (For instance, you could use this fact to "prove" 1 = 0!)

You could help your learners to slowly build their understanding of limits, by using examples of functions that they have already dealt with. For example, let us go back to Example 2, in Unit 4. In that example, s is the function describing the distance of the truck from its starting point at time t given by

$$s(t) = \begin{cases} 50t, & 0 \le t \le 30/60 \\ 40 - 30t, & 30/60 \le t \le 40/60 \\ 20, & 40/60 \le t \le 45/60 \\ 70t - 32.5, & 45/60 \le t \le 90/60 \end{cases}$$

The students should draw the graph of the function (as shown in Fig. 9, Unit 4). With the help of the graph, they could try to calculate the limit of the function, say, at the point $t = \frac{30}{60}\left(= \frac{1}{2}\right)$. What they need to understand is that it is the value that s(t) **approaches** as t moves closer and closer to $\frac{1}{2}$ from the left or from the right. In this case they see that it is 25, which is also $s\left(\frac{1}{2}\right)$.

Similarly, they could find the limit of the function at several other points also.

However, **a misunderstanding may students have is that the limit of the function at a point is the value of the function at that point**. This is probably because all the examples we expose them to seem to be of continuous functions, for which this is true. In the following example, we see how one teacher deals with this misconception.

Example 1 : A school teacher, Amrita, was introducing the concept of limits to the students. She had given them examples of some polynomial functions, and asked them to find the limits at certain points of their domains. However, when she asked the students what $\lim_{x \to 1} f(x)$ was, where $f : \mathbf{R} \to \mathbf{R}$: $f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$, most of the students answered '2'. On asking them their reasons, she realised that for them the limit was the value of the function at the point, i.e., f(1).

In order to remove this misunderstanding from the minds of the students, she decided to give them real-life problems involving discontinuities, like the one given below.

A three-wheeled scooter charges a minimum of Rs.5/- for a distance upto the first kilometre travelled, and Rs. 2/- for every kilometre (or part of it) travelled after the first one. Define the cost function, C, and draw its graph. Also find its limit at s = 3 and s = 4.5, where s denotes the distance covered.

The students were divided up into small groups, and asked to discuss the problem with each other and give a solution. As she went around the groups, she found that each group had defined C in different ways as:

$C(x) = 5x + 2$; or
$C(x) = 2x$, $C(1) = 5$; or
$C(x) = 5 + 2x$.

Only two children in one group were arguing with the others about the following representation they had got:

$$C(s) = \begin{cases} 5, 0 < s \leq 1 \\ 5 + 2, 1 < s \leq 2 \\ 5 + 4, 2 < s \leq 3 \\ \vdots \\ 5 + 2n, n < s \leq n + 1. \end{cases}$$

Amrita decided to involve the whole class in this group's discussion. Amrita asked the two children to explain their reasons for defining the function in this way. They did so, very cogently. When the other students were asked for their reactions, they agreed with this definition.

However, one child asked the teacher if this function couldn't be defined in one line. Amrita asked everyone to think about this. There was quite a bit of discussion on this, with students trying to see if the way they had defined it would work. Finally, after quite a bit of argument and modifications, the students agreed to the representation

$C(s) = 5+2n, \quad n < s \leq n+1 \ \forall\, n = 0,1,2,\ldots$

Now came the question of graphing the function. Some students volunteered to do this on the board. Amrita got three kinds of diagrams from them (see Fig. 2).

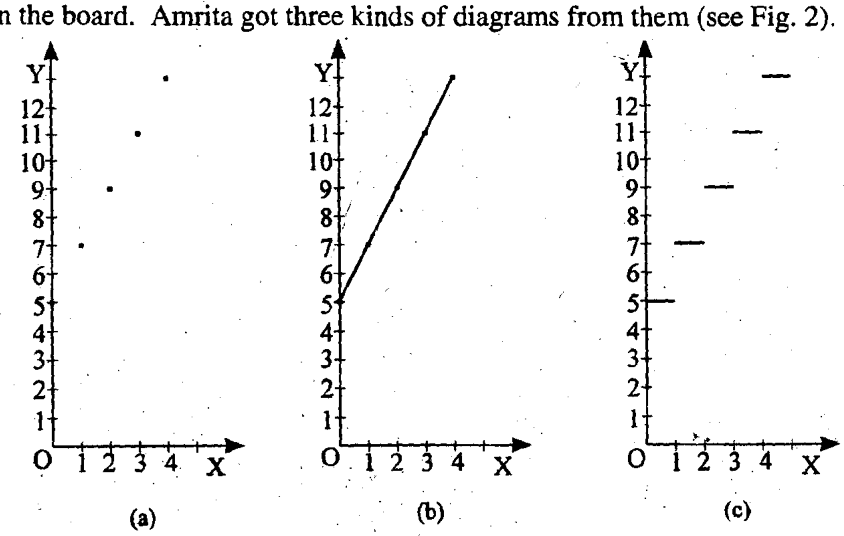

Fig. 2

She asked the students to explain why they thought their graphs were right. Each explanation generated a discussion. But, finally, Fig. 2(c) was accepted.

Now came the part about the limit. Amrita, again, asked the students to discuss this in their groups. After about 10 minutes, she called a halt to their interchanges, and asked the groups to come up with what they had done, one by one.

The first group's representative, Rajan, came up to the board, and started with the calculations. He wrote

$\lim\limits_{s \to 3} C(s) = \lim\limits_{s \to 3} 5 + 2n = 5 + 2 \times 3 = \underline{11}$.

At once another child, Sukriti, objected to this. She said it should be $5+2\times2$, because this is the value at $s = 3$. The teacher asked other students to give their views. Many students supported Rajan, but there were a few who thought that Sukriti's calculations were correct.

Now Amrita decided to ask them to look at the graph and tell her what happens to C(s) as s approaches 3 from the left. She moved her finger along the graph, so that they saw that C(s) gets closer and closer to 9, and hence the left hand limit at $s = 3$ is 9. So it was agreed that $\lim\limits_{s \to 3^-} C(s) = 9$.

Now, Amrita asked for a volunteer to find the right hand limit at the same point. Shalu came up to the board and traced the graph with her finger as she had seen the teacher do, and said 11. Here the other students started muttering loudly that this was wrong. It should be 9, because $C(3) = 9$. This made Shalu look uncertainly at the teacher. So, Amrita stepped in and took this opportunity to point out the difference between the limiting value and the value at the point.

She reminded them that in the earlier examples it so happened that the two coincided. But this need not be so, as in this case. Here, the function had taken a jump at $s = 3$, as she showed them from the graph. So, the limit from the right was different from C(3). In fact, it was 11, as Shalu had said. "Since the limit of the function calculated from different directions are two different numbers, the limit of C(s) at $s = 3$ **does not exist**," she told them.

Now Amrita asked the students to give her the limit at $s = 4.5$. The children managed to do this part easily, since there was no 'jump' at this point.

Finally, Amrita took up the following problem for discussion in groups.

Find the limit of the function g defined by $g(x) = \begin{cases} x, & x < 0 \\ x^2, & 0 \le x \le 2 \\ 3x, & x > 2 \end{cases}$

at the point $x = 0$ *and* $x = 2$.

After a moments hush, a lot of murmuring erupted in the class. The teacher helped the students to focus their talk into meaningful discussions. She participated in the group discussions, sharing their ideas, asking questions, etc. After 10 minutes, she asked the children to share the results of their discussion with everybody.

She asked the students of Group A to explain to the others what happens to g(x) when x moves closer and closer to 0 from the left. A child from the group came to the board, drew the graph of g, and explained why the value of g(x) also gets closer to 0. Therefore, everyone agreed, the limit of g(x) as x tends to 0 from the left hand side is 0. On similar lines, the students of Group B explained that as x moves towards 0 from 2, g(x) also moves towards 0.

Now she asked the children if this meant that the limit existed or didn't exist as x approached 0. The children agreed that it did exist because it was the same from both the directions.

Again, to find the limit at $x = 2$, she asked a Group C child to come forward. He explained why $\lim_{x\to 2^-} g(x) = 4$.

But, for calculating the right hand limit at $x = 2$, there was lots of confusion in the class. None of the groups wanted to come forward. So the teacher started again by asking the students relevant questions. She proceeded by asking them to find g(x) for $x = 2.5, 2.25, 2.1, 2.01$. What was happening to g(x)? The students observed that it was decreasing, coming closer and closer to 6. "So, can we say that $\lim_{x\to 2^+} g(x) = 6$?", she asked. They agreed, and added that like in the last problem, the limit of the function as x tends to 2 does not exist.

This remark made Amrita feel that the concept had taken root somewhere, connections had been made, and now she had to build on this in the next few classes.

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In the example above, the teacher gave the learners many opportunities to apply their minds on the following points to remove misunderstandings regarding 'limit'.

- The limit of the function need not always exist.
- The value of the function at a point may not be equal to the limit of the function at that point.

You may now like to do the following exercises with your learners.

E3) Draw the graph of the function f defined by $f(x) = \dfrac{x^2 - 6x + 8}{x - 2}$, and evaluate its limit as x tends to 2.

E4) Give your students a function that oscillates at $x = 2$ to graph. Ask them to find the limit at $x = 2$, if it exists. What are their reasons for their answers?

A result that we use very often for finding limits is the 'sandwich' (or 'squeeze') theorem. Many students do not know why it is called this, and/or why it works. A pictorial representation of its utility may greatly help in this matter. For example, you could ask them to consider the graph of the function $f(x) = \sin\frac{1}{x}$ (see Fig.3).

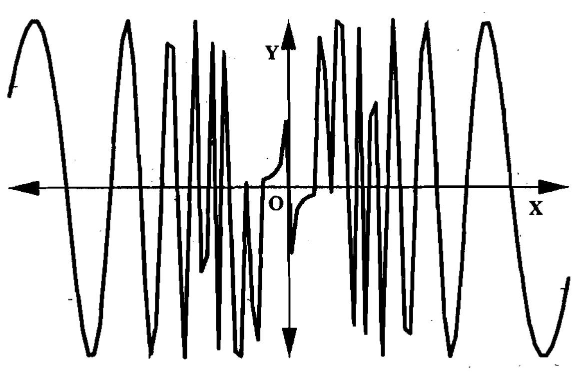

Fig. 3 : Graph of $y = \sin \frac{1}{x}$

Looking at it, can they evaluate the limit of the function as x tends to 0? Now give them the graph of $g(x) = x^2 \sin\frac{1}{x}$ (see Fig. 4), and ask them to compare the behaviour

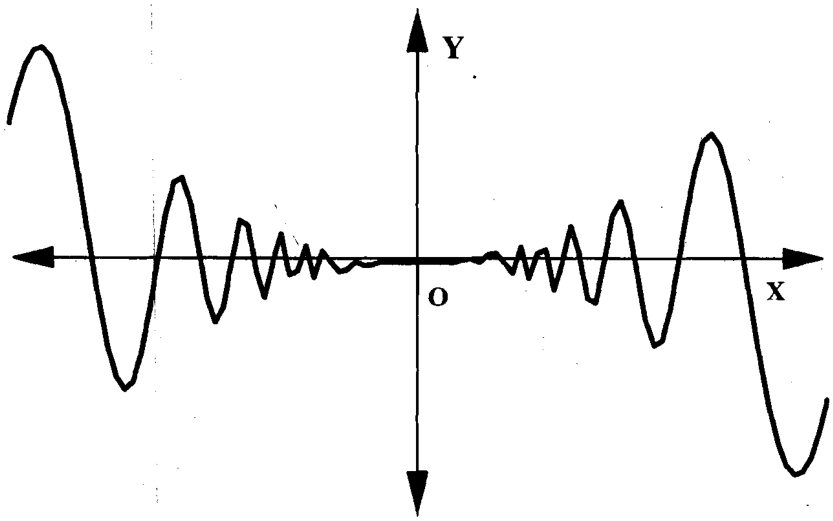

Fig. 4 : Graph of $y = x^2 \sin\frac{1}{x}$

of f and g in any neighbourhood of 0. Do they see that as x gets closer to 0, the function oscillates between -1 and $+1$? So, f fails to have a limit as $x \to 0$.

Now, you can lead them from f to g. Let them graphically (as in Fig. 5) see that for any real number x, $-x^2 \leq x^2 \sin {}^{1}\!/_{x} \leq x^2$. Do they see that as x gets closer to 0,

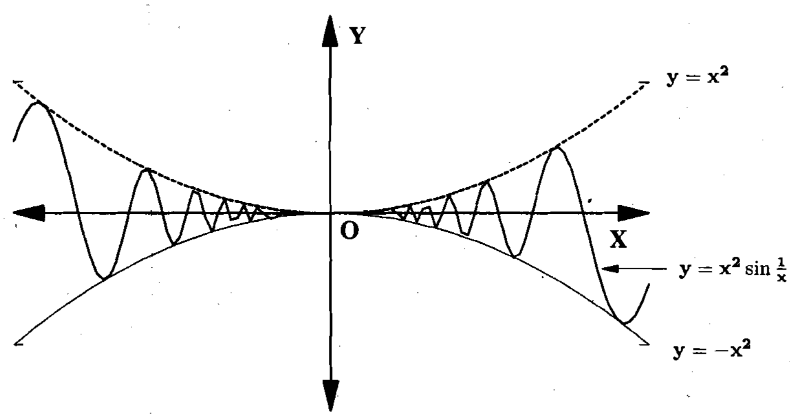

Fig. 5 : $y = x^2$ and $y = x^2$ 'squeeze' $y = x^2 \sin\frac{1}{x}$.

x^2 and $-x^2$ become very small in magnitude? Therefore, any number in between becomes very small in magnitude. So, they see that x^2 and $-x^2$ 'squeeze' $x^2 \sin 1/x$, forcing it to behave like they do. This is why

$$\lim_{x\to 0} x^2 \sin {}^{1}\!/_{x} = \lim_{x\to 0} x^2 = \lim_{x\to 0} (-x^2) = 0.$$

Also try the following exercise with your learners.

E5) Use the sandwich theorem to evaluate the limit, $\lim_{x\to 0} x \cos\frac{\pi}{x}$, if it exists.

One of the difficulties students face is dealing with infinity. And, this is compounded when they are required to deal with limits involving infinity. The students can be greatly helped in this matter with a **visualisation** of how a function behaves at larger and larger values of its domain.

How do we explain to them what a limit is as $x \to \infty$? Since ∞ is not a real number, we cannot describe closeness to ∞ in terms of intervals around ∞, as there is nothing to the right of ∞. But we can describe the closeness in terms of open intervals of the form $]k,\infty[= \{x \in \mathbf{R} : x > k\}$. Clearly, the larger k is, the 'closer' we are to ∞. So, we can help them interpret $\lim\limits_{x\to\infty} f(x) = L$ to mean that f(x) can be brought arbitrarily close to L, provided x is sufficiently close to ∞, i.e., f(x) can be brought arbitrarily close to L, provided x is **sufficiently large**.

Diagrammatically, we can show them the asymptotes, as in Fig. 6, to make the point.

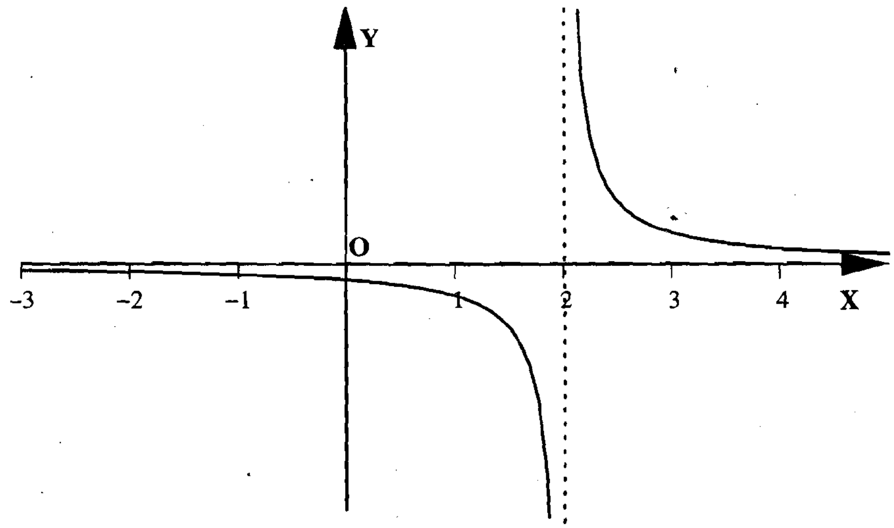

Fig. 6 : Graph of f(x) = $\frac{1}{x-2}$, $x \in \mathbf{R} \setminus \{2\}$.

You could ask your students to study the graph and discuss what happens to f(x) as $x \to -\infty$ or as $x \to \infty$.

Using examples as shown in Fig. 6, you could also help the students to clarify their understanding of infinite limits. Here, the student needs to understand that the statement '$\lim\limits_{x\to 2} f(x) = \infty$.' is really saying that 'as x gets closer to 2, the function f(x) becomes larger and larger.' **Since '∞' is not a number, we cannot say that the limit exists**, in the way we can for a number by taking a neighbourhood of the number. The students need to understand (and see) that as x approaches 2, $\frac{1}{x-2}$ becomes arbitrarily large, and it can't stay close to any finite number L. So, in effect, $\frac{1}{x-2}$ **has no limit as x approaches 2.**

Another example to make the point could be of a polynomial function of degree 1 or greater, which will eventually take off to infinity as x tends to ∞ or $-\infty$.

Why don't you try an exercise with your learners now?

E6) Give the students some examples of functions to explore the limits to infinity of a function and draw conclusions about vertical and horizontal asymptotes. What kind of errors did they make? What was their reasoning behind these errors?

A major activity that may help the students to become comfortable with 'infinity' is given below. In fact, doing this activity can help in reinforcing several concepts and processes like learning about fractals, inductive reasoning, spatial understanding, etc.

Activity (Generating Fractals)

In this activity, students are introduced to a method of generating fractal curves. They apply this method 'by hand', using pencils and graph paper to generate the first few iterations of a fractal curve. Various concepts about the curve's properties are investigated. 'What if' questions lead to discovering patterns, relationships, and investigating the idea of a limit involving infinity.

Materials : Graph paper, pencils, the computer programme application "Snowflakes" (if a computer on which to run this application is available).

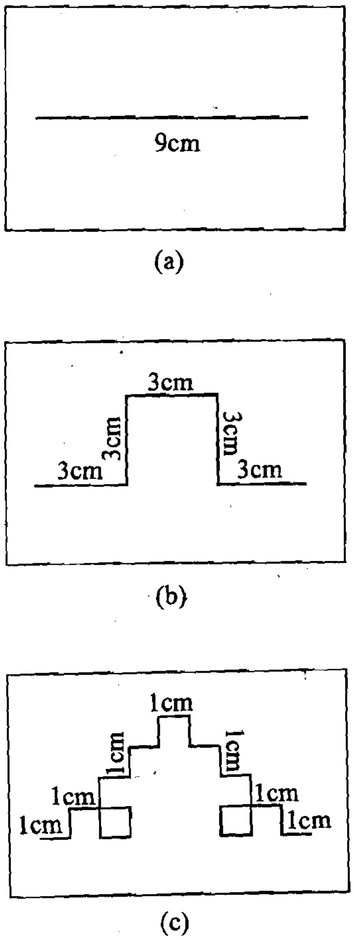

Fig. 7

Steps in the Activity

1. Ask each student to draw a horizontal line segment 9 units long on the graph paper (see Fig. 7(a)). (For convenience, let one unit be equal to the length of a side of one square on the graph paper.) After they go through a couple of more steps, the students need to be asked why they started with 9 units, and not 5 (say).

2. Next, the students should divide this segment into three equal parts, and replace the middle part by three segments, each of length 3 units — moving vertically upwards 3 units, then rightwards 3 more units, then downwards 3 units. In effect, they place three sides of a square with side 3 units instead of the middle segment (see Fig. 7(b)) on the middle of the original line segment.

3. Repeat Step 2 with each of the five segments obtained in Step 2. That is, the middle third of each segment from the first iteration is replaced with a similar square bump (see Fig. 7(c)).

4. And continue repeating Step 2 with each segment formed.

Each step should be drawn separately.

To help the students focus on the point you want them to, you could do the following:

After, say, 4 iterations, you could ask them how many line segments they drew at each step. Do they see a pattern emerging in the number of line segments at each iteration? How many line segments would it take to draw the next iteration? You can ask similar questions about the length of the curve at each iteration. In fact, ask them to make a table like the following one. This would be helpful in organizing the information, as well as helping the students identify patterns.

Iteration	Number of segments	Length of each segment	Total length
0	1	9	9
1	5	3	15
2	25	1	25
3	125	1/3	125/3
4	625	1/9	625/9

Each of the last three columns are geometric sequences. You could encourage your students to discover this themselves, in groups, or as a class. The last column in particular can be useful as an exercise in number sense. The following are some possible discussion questions.

- What will the next row of the table look like?
- What patterns do you notice in the columns?

- What would the tenth row of the table look like?
- What is happening to the total length? How much is the length changing each time? (It changes by a different amount each time. The amount it changes by is increasing.)
- How else might we quantify the rate of increase? (If they got the patterns for the first two columns, you might help them see that they increase and decrease, respectively, by a constant ratio. How might they figure out if that is happening in the third column? Some students may need a hint like "Try dividing length one by length zero, and length two by length one. Do you notice anything?")
- If you want to extend the students further, once they work out what the ratio is for the third column, you could ask them to discuss what would happen if it were 3/5 instead of 5/3, etc.

There are computer programmes like 'Snowflake' that students can use also to see how the curve changes with each iteration. Such a programme very easily shows them several interactions.

To help them get used to 'limit as $x \to \infty$', ask the students what would happen eventually to the area under the curve if we kept increasing the number of iterations. They can experiment by increasing the iterations. What do they notice? Will the total area under the curve and above the line in Step 1 keep getting bigger (see Fig. 8)?

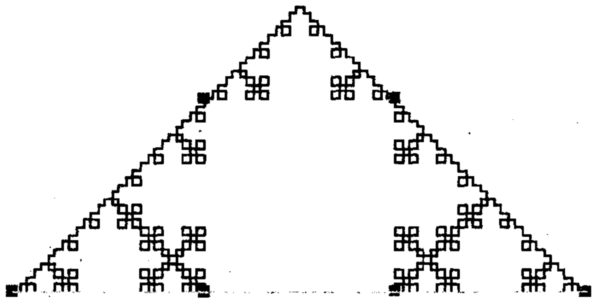

Fig. 8 : Last iteration using 'Snowflake' in which the shape of the curve is still clearly visible

The area will continue to increase, but it increases by a smaller amount each time. Notice how, as the iterations increase, the overall shape of the pattern seems to be "filling" a triangle. What is the area of that triangle? Will the area bounded by the curve ever get bigger than the area of the triangle?

Further discussions in this activity include:

- Calculating the difference between the area bounded by the curve and the area of the limiting triangle by drawing the triangle around the various iterations and adding up the area of the remaining spaces.

- Discussing the fact that the length of the curve goes to infinity as the number of iterations goes to infinity.

Here's a related exercise now.

E7) Design an activity for your learners with the objective of helping them develop an understanding of $\lim_{x \to \infty} f(x) = L$. Try it out with your learners, and note down their reactions and other outcomes.

Let us now discuss ways of connecting the understanding of limits you would have developed in your learners with the idea of continuity.

5.3 INTRODUCING DISCONTINUITY

By the time you want to introduce your students to the notion of continuity, they would be familiar with many kinds of functions and their graphs. While studying 'limits' they would have come across functions that have gaps, breaks or jumps in their graphs, as well as those that don't have such features. You can use these examples and non-examples to introduce them to the idea of continuity.

As we have often suggested throughout the course, a good way of helping the learner to learn a concept is to allow her to construct her own understanding of it. The following situation may interest you, in this regard.

Example 2 : The teacher, Kumar, was quite confident about his students' understanding of limits. A few of them also seemed to be familiar with 'continuous functions', as he discovered when he chatted with them. However, he wanted to introduce all the students to this concept. So he drew several figures on the board (see Fig. 9), and told the children, "This (*pointing to (b)*) is a continuous function, and these (*pointing to (a) and (c)*) are not continuous. Why do you think I have called (b) continuous? What is the property it has that the other two don't have?"

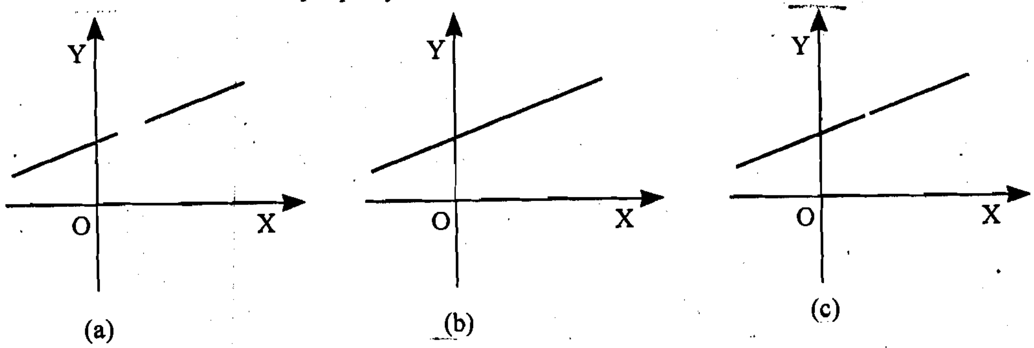

Fig. 9

After looking at the graphs several students spoke out together. Kumar asked one of them, Kamla, to share her understanding with the others. She said that 'continuous' means one line. The middle one is a line, while the others are two lines.

Here you need to be very careful in your choice of examples and non-examples of a concept. Otherwise some other unintended common features may show up which are not part of the definition of the concept.

Here is when Kumar realised that unwittingly he had given diagrams that were giving some children misconcepts. So, he immediately drew some more diagrams (see Fig. 10), and again pointed out the continuous ones.

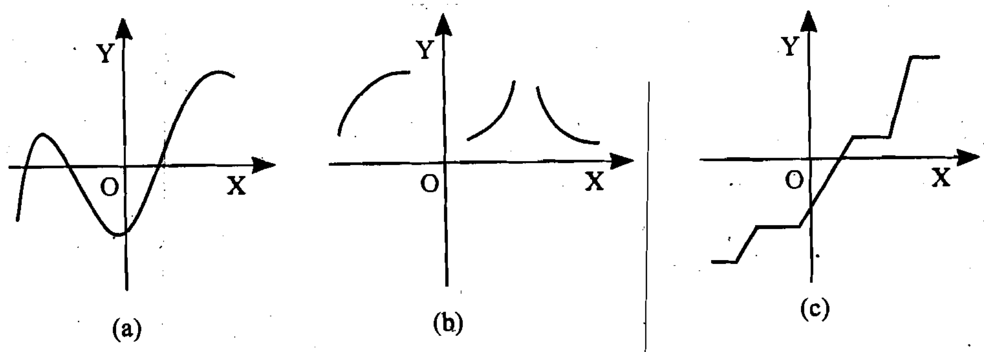

Fig. 10

Again he asked the children the same question. Through some discussion, the children concluded that "if there is only one curve, with one piece, then it is a continuous function. If the curve is in many pieces, then it is not continuous."

One student yelled out that it is a graph drawn without lifting pencil from paper. Using such reactions, Kumar gradually led his learners to the formal definition of continuity of a function. He also helped the students realize how they needed to use their knowledge of limits studied earlier to proceed with checking a function for continuity or discontinuity.

As a next step, he asked the students to divide up into groups. Each group was given a function to graph and find, geometrically as well as algebraically, whether it was continuous at a point.

Through such exercises, and full class discussions of the solutions, the students realized that the continuity of a function at a point a means that f(x) gets closer and closer to f(a) as x gets closer and closer to a, i.e. , the function moves continuously towards its actual value at a as x moves towards a. Hence the word "continuous".

Kumar ended the session with asking the children to note down, as homework, where they see continuous curves and discontinuous curves in their homes or outside.

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These types of activities help the students to learn the subject in an informal manner. This helps in interesting them in the subject.

The important point that you need to help your learners realise during your interaction with them is that there are **two numbers** we consider when we consider $\lim\limits_{x \to a} f(x)$.
One is the number that f(x) is getting closer and closer to as x moves towards a. The other is the value of the function at the point a. **These two numbers are distinct** unless the function is continuous at x = a.

Also, the students need to realise that the symbol f(a) tells us nothing about the function at any point other than a. Therefore, **the two expressions $\lim\limits_{x \to a} f(x)$ and f(a) are independent of each other**. The students need to understand that the value of one has no bearing on the value of the other.

Here's an exercise about this now.

E8) List at least two examples each to illustrate the following possibilities to your learners, where f is a function.

i) $\lim\limits_{x \to a} f(x)$ exists but f is not defined at a.

ii) f(a) is defined, but $\lim\limits_{x \to a} f(x)$ does not exist.

iii) Both $\lim\limits_{x \to a} f(x)$ and f(a) exist, but are not equal.

iv) $\lim\limits_{x \to a} f(x)$ exists and equals f(a)

v) Neither $\lim\limits_{x \to a} f(x)$ nor f(a) are defined.

Hand in hand with 'continuity' goes 'discontinuity'. The different situations in which a function is discontinuous have been suggested in E8. Your students could be familiarised with different kinds of discontinuities like 'jump' and 'removable', along with explanations for why these names are used. Appropriately chosen examples would help you to make your point.

For instance, you could tell them why $x = -1$ is a removable discontinuity of f defined by $f(x) = \dfrac{4x+4}{x^2-1}$ $\forall x \in \mathbf{R} \setminus \{-1,1\}$. Ask them to draw the graph (see Fig. 11) and

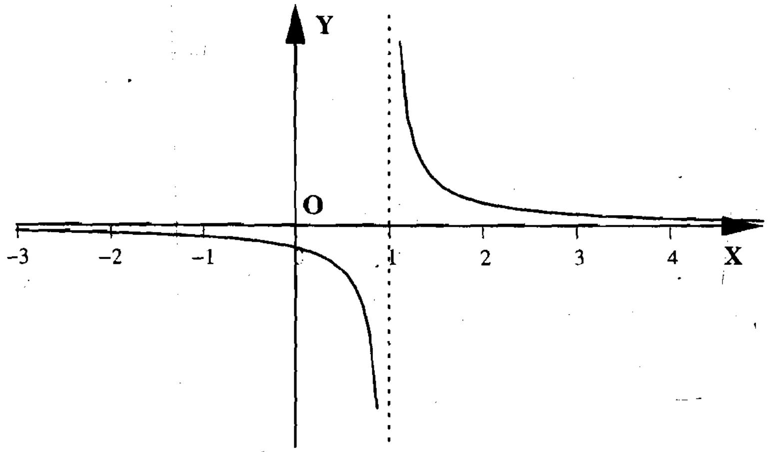


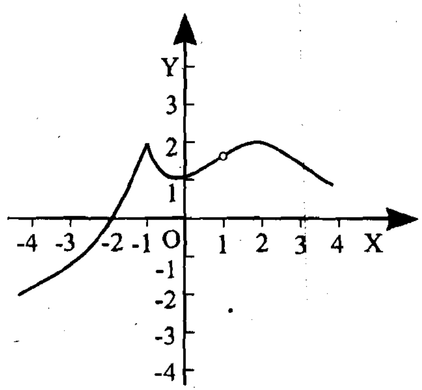
Fig. 11 : Graph of $\dfrac{4x+4}{x^2-1}, x \neq \pm 1.$

observe what happens as $x \to -1$. Do they see the small gap or hole at $x = -1$? Do they realise that this means that $f(-1)$ does not exist, but the function approaches -2 as x gets closer and closer to -1. In other words, the limit of the function as x tends to -1 exists and is equal to -2, but $f(-1)$ does not exist. However, if we define f at $x = -1$, by $f(-1) = -2$, then this point of discontinuity would be removed. This is why $x = -1$ is a **removable** discontinuity.

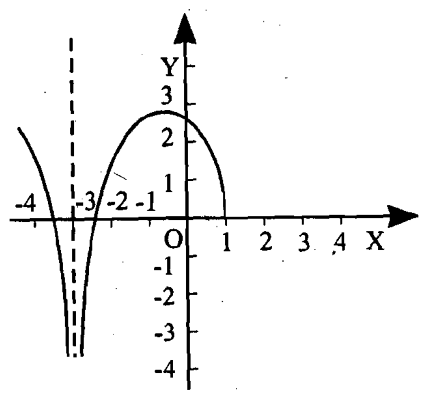
A question may arise in the mind of the students — can all discontinuous functions be made continuous in this way? Doing the following exercises will help you to explain to them why the answer is 'No'.

E9) Give an example to your learners that would clarify to them what a 'jump discontinuity' is.

E10) Determine whether the following functions are continuous at the points $x = -3$ and $x = 1$. At the discontinuous points, indicate which condition of continuity does not hold.



(a)



(b)

Fig. 12

E11) i) Design an activity for your learners to help them improve their understanding of 'discontinuity'.

ii) What questions did you keep in mind while designing it?

iii) To what extent did the activity **not** achieve its aim, when you tried it out with your learners?

E10(b) leads me to something students have often asked me about, namely, continuity over a closed interval. For instance, they consider the function $f(x) = \sqrt{a-x}$, defined in the interval $[a, \infty[$, and say that at $x = a$, only the right-hand limit $\lim_{x \to a^+} f(x)$ exists. So, according to the definition of continuity at a point, they say f is not continuous at x = a. But then that means that no function can be continuous at the endpoints of its domain. So, you need to explain that we relax the requirement of continuity when it comes to the endpoints and redefine it in the most obvious way, namely,

a function **f is continuous in [a,b]** if

i) it is defined on [a,b],

ii) $\lim_{x \to c} f(x) = f(c)\ \forall c \in]a,b[$,

iii) $\lim_{x \to a^+} f(x) = f(a)$, i.e., the right-hand limit at a agrees with the function value at a,

iv) $\lim_{x \to b^-} f(x) = f(b)$, i.e., the left-hand limit at b agrees with the function value at b.

As always, these notions are understood better if the students work on a variety of problems involving them. Here are some related exercises for you to do with your learners.

E12) Does there exist a function which is discontinuous everywhere? If yes, give an example.

E13) Discuss the continuity of $f(\theta) = \sec \theta$ for $0 \le \theta < 2\pi$..
(Here the students should be trained to understand what 'discuss' means in this context. They need to explain why f is continuous at **all** the points, except for the points of discontinuity.)

So far we have looked at how to expose learners to different kinds of discontinuity, and to continuity. In the end let us take a quick look at what we have covered in this unit.

5.4 SUMMARY

The major points we have focussed on in this unit are given below.

1. We looked at some problems students face, and errors they make, when dealing with 'limits'.

2. We discussed some methods of introducing students to limits so that their understanding develops 'naturally'.

3. In particular, we considered some activities for helping the learners understand 'infinite limits' and 'limits at infinity'.

4. We discussed some strategies for removing students misconceptions regarding 'continuity' and 'discontinuity'.

5.5 COMMENTS ON EXERCISES

E1) Some limits are measurable, and some aren't. For instance, don't we often hear the term 'the height of crudity' or 'the limits of my patience'?

Regarding measurable limits, there is, for instance, an upper limit to the amount of rice a given student can eat at one meal. Think of other examples that your students can relate to.

E2) From high school geometry, you would recall the result:

If an n-sided regular polygon is inscribed in a circle of radius 1, and a second circle is inscribed in the polygon, then the inner circle has radius cos (π/n).

In this pattern, the radii of the circles are
$\cos \pi/3$, $\cos^2(\pi/3)$, $\cos^3(\pi/3)$, ...
Hence, the radius of the n^{th} circle is $\cos^n(\pi/3)$, i.e., $(1/2)^n$.
As n tends to infinity, the resulting radius tends to 0.

What were the other patterns/relationships that your students found? Did they think of other extensions of this exercise? If so, what were they?

E3) Ask your student to look at the graph of the function (see Fig.13).

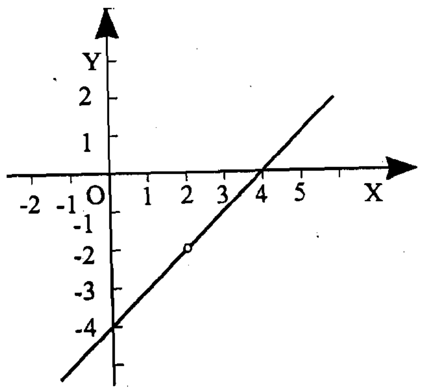


Fig. 13

Do they observe any holes in the curve? What happens at $x = 2$? The value of the function at $x = 2$ is not defined. However, the function value **approaches** -2. Therefore, the limit of the function as x tends to 2 is -2.

E4) For instance, you may take the function $f(x) = \cos\left(\frac{1}{x-2}\right)$.

E5) $f(x) = \cos \pi/x$ is an oscillating function and it oscillates between -1 and $+1$. What happens in the case of $x \cos \pi/x$? The students need to consider both the cases, namely, $x > 0$ and $x < 0$.
Let them draw the graphs in both cases and see what happens (Fig. 14).

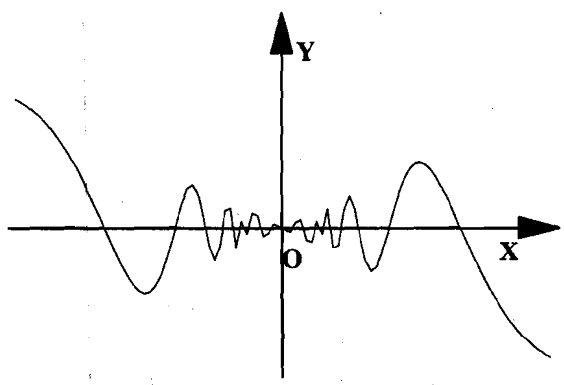


Fig. 14 : Graph of $y = x \cos \pi/x$

i) $x > 0$: $-1 \leq \cos \pi/x \leq +1$. Therefore, $-x \leq x \cos \pi/x \leq x$.

By the sandwich theorem, we find that $\lim_{x \to 0} x \cos\frac{\pi}{x} = 0$

ii) $x < 0$: They can proceed as before to find that $\lim_{x \to 0} x \cos\frac{\pi}{x}$ is the same, namely, 0.

E6) For instance, you could take the functions f defined by

$f(x) = \frac{1}{x-3}$ or $f(x) = \frac{e^x}{e^x+1}$, or $f(x) = \sqrt{x^2}$.

How did they go about the task? Did they observe that $y = 0$ and $y = 1$ are horizontal asymptotes for the second function?

A common mistake regarding the third function often made by the students is that they take $\sqrt{x^2} = x$, not $\pm x$. So, they ignore the case when $x < 0$. Record other errors they make.

E7) For instance, ask them to draw regular polygons of more and more sides. Do they note that the n-gon tends to a circle in the limiting case as $n \to \infty$? The students can also be asked to work on it on a computer where this can be clearly seen.

E8) It will be helpful for your learners if you give them the geometric and algebraic representations of the less simple functions.

E9) Consider the function $f(x) = \begin{cases} x^2+5 & x<-1 \\ 6x & x=-1 \\ 2x+3 & x>-1 \end{cases}$. Check the function for continuity at $x = -1$. Using its graph, explain why it is a 'jump' discontinuity. There is no way in which the function can be redefined at $x = -1$ so that it becomes continuous.

E10) (a) The function is continuous at $x = -1$, but not at $x = 1$ as there is a small gap there, i.e., the function is not defined at $x = 1$. So, $x = 1$ is a removable discontinuity.

(b) $x = -3$ is an asymptote. The function has a limit as $x \to -3$, but the function is not defined at $x = -3$. Hence the function is not continuous at $x = -3$. This is a non-removable discontinuity. At $x = 1$, it is continuous, since $\lim_{x \to 1^-} f(x) = f(1)$.

E11) Your design would depend on your answer to (ii). Questions like the following need to be answered.

- What is its objective?
- What is the nature and quality of the students' participation?
- What materials are required?
- How much time is required?

You may choose to just give a variety of exercises, or a game that can draw the whole class into an animated discussion on the concept, or another activity.

E12) For example, $f(x) = \begin{cases} +1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

Ask your students to pick any point in the interval, say x_0. Ask them to find $\lim_{x \to x_0} f(x)$.

How did the students go about it? Did you allow a peer group discussion to take place? What kind of conceptual errors showed up in such discussions?

Upon **reflection**, in what way has this **helped you assess your teaching strategy**? Please record all these points in **your logbook**.

E13) Let us draw the graph of $f(\theta) = \sec\theta$ for $\theta \in [0, \infty[$ (see Fig. 15). The students should be able to explain why f is not continuous at $\theta = \pi/2$ or $3\pi/2$. They should also explain why it is continuous at every other point in the interval.

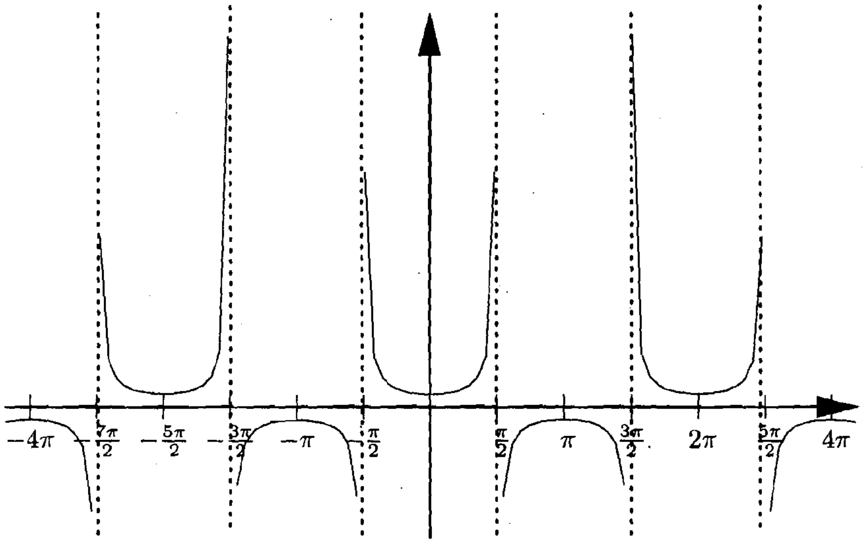


Fig. 15

UNIT 6 LOOKING AT THE DERIVATIVE

Structure

Page Nos.

6.1 INTRODUCTION

In the last two units we focussed on concepts that are pre-requisites for studying calculus. In this unit, we deal with problems students face in understanding differentiability. As we have said earlier, the understanding of 'derivative' is dependent on how well 'limit' has been understood by the learner. Therefore, much of what has been discussed in the previous unit, is relevant to the issues taken up in the present unit.

We begin this unit with suggesting ways of helping students become comfortable with the notion of differentiation. Here, a remark made by an NVS teacher is relevant. He said, "My students can differentiate most functions, but they do it mechanically. They don't understand the underlying process." In Sec. 6.2, we look at ways of helping students assimilate this underlying process.

There are several misconceptions regarding when a function is differentiable and when it is continuous. In Sec. 6.3, we look at some examples that may help in removing these misconceptions.

Finally, in Sec. 6.4, we take up the use of the derivative for curve tracing and analysing the behaviour of a function. Students often miss the significance of the first and second derivatives in this context. They also get confused between concepts like 'critical (or stationary) point' and 'point of inflection'. We have suggested an approach that has been found helpful by some teachers in alleviating this problem.

As in the previous units, this unit will be of use only if you actually try the exercises and activities with your learners. You must follow this up with analysing the students' responses. You could then alter your strategies, if necessary, based on the assessment you make.

Here, now, is an explicit list of the broad objectives of this unit.

Objectives

After studying this unit, you should be able to develop the ability of your learners to

- explain the meaning of 'derivative', and give its geometric and physical interpretation;
- explain when a continuous function need not be differentiable;
- use the first and second derivatives of a differentiable function for analysing its behaviour and for tracing its graph.

6.2 WHAT IS DIFFERENTIATION?

Let me start by bringing you a quote from an article by Judith Grabiner in Mathematics Magazine, Vol. 56, 1983. She writes :

"Historically speaking, there were four steps in the development of today's concept of the derivative, which I list here in chronological order. The derivative was ***first used****; it was* ***then discovered****; it was* ***then explored and developed****; and it was* ***finally defined****. That is, examples of what we now recognize as derivatives were first used on an ad hoc basis in solving particular problems; then the general concept lying behind these uses was identified (as part of the invention of calculus); then many properties of the derivative were explained and developed in applications to mathematics and to physics; and finally, a rigorous definition was given and the concept of derivative was embedded in a rigorous theory."*

Please think about this order carefully. Is it the same order in which we expose our learners to the derivative? If not, shouldn't we help our learners to arrive at the concept in a more intuitive and less formal manner, as it has been developed historically? Because we don't do this, most students react to the question given in the title of this section with the response. "It is d by dx." If probed further, they give an example like $\frac{d}{dx}(x^n) = nx^{n-1}$, or some other examples. Very rarely do I find a student who tries to explain its mathematical meaning, or its geometric or physical interpretation.

Even though most teachers usually tell students that the derivative gives the slope of the tangent to a curve, and draw a diagram to explain it, students don't understand what is happening. Even students in college have a problem with this. Maybe drawing a diagram, giving some solved examples and many practice examples as homework is **not** the way to help the learner learn the concept. We need to present this aspect, as well as differentiation as a measure of the instantaneous rate of change, differently.

In this context, it would be useful to note that students even have problems with understanding the difference between average and instantaneous rates of change. How would you address this problem? Rather than explaining this mathematically, you could give them examples of, say, movement of a vehicle.

To start with, it may be useful, to clarify the student's understanding of 'average rate of change'. You could ask your students what it means if a car travelled from Ajmer to Jaipur with an average speed of 50 km/hr. Does this mean that at each point of time during the journey its speed was 50? What about the traffic light being red on the way? And the time the car was suddenly stopped by a truck driver?

In fact, students make other mistakes regarding this concept. For instance, ask them the following problem:

If you travel at a speed of 40 km./hr. going from Cochin to Trichur, and at a speed of 60 km./hr. coming back, what is the average speed for the round trip?

Did you get the usual response I get from many students to such a question, namely, 50 km./hr.? This happens because the students don't think about what 'average speed' really is. Here, you would need to give them some hints to work out why the average speed is 48 km/hr. In fact, if the students used their common sense, they would realise that more time is spent while going at 40 km./hr. than is spent coming back the same distance at 60 km./hr.

Now, how would you help your students to think about the instantaneous rate of change as the average rate of change **at that particular instant?** They need to see that you are making your time interval smaller and smaller, and looking at the average value of the function over this smaller and smaller interval. This average is $\frac{f(t_0+h)-f(t_0)}{(t_0+h)-t_0}$. As h gets smaller and smaller, nearer and nearer to zero, the limiting value of this quotient, if it exists, is what we call the instantaneous rate of change of f with respect to t at the instant t_0. In fact, one teacher put it to her students very nicely, as given in the following example:

Example 1 : Ms. Grace regularly coaches senior students in mathematics. A question she is frequently asked by her students is 'How is the derivative useful?' She usually responds by explaining, "Imagine you go on a car ride. Suppose you know your position at all times. In other words, at 8 a.m. you are in the garage, at 8 a.m. and 5 seconds you are just outside the garage, at 8 a.m. and 10 seconds you are on the road just in front of your house, and so on. At every moment during this ride, your speedometer showed the speed of your car. So, if you knew your position at all times, at the end of your trip can you work out what your speedometer showed at any particular instant of time? The answer is, yes, you can. The derivative provides a method for doing this.

She goes on to give them the simplest situation where one can compute what the speedometer reading is, that is, driving at the same speed over the entire distance. In this case, of course, the students do conclude that if you drive 50 kilometres in one hour throughout at the same speed, then your speedometer read 50 km. per hour throughout the trip.

In the situation where the car is driven at different speeds, Grace tries to get the students to consider the whole trip as made up of several short trips, say, one trip involving taking the car out of the garage, another trip would be driving the car onto the road, and so on. "Over each of these tiny trips, your speed doesn't change much", she tells them, "So, you can pretend that your speed didn't change at all. So, you know how to compute the speed for each tiny trip. This gives you a good idea of what your speedometer read for that part of the big trip. But, remember, the assumption that the speed didn't change over each tiny trip is generally wrong, and so you only get **an approximation** to the correct answer. But, the main idea behind the derivative is that the smaller you make the tiny trips used in your computation, the more accurately you will be able to compute the actual speedometer reading." In this way Grace tries to help the students understand the average change in an infinitesimal time interval.

Sometimes Grace explains the derivative to the 'Commerce Stream' students in the following way:

"When you see the sensex (sensitivity index) report of the stock market on the TV in the evening, it measures the change in the aggregate stock index per unit of time. This is best understood visually by the slope of the graph joining the various points showing the stock index (see Fig. 1). If we measure the closing stock price from one day to the next, we notice that the graph gets higher on days when the price change is positive, and lower when the stocks go down. **The steeper the slope, the faster the change.**

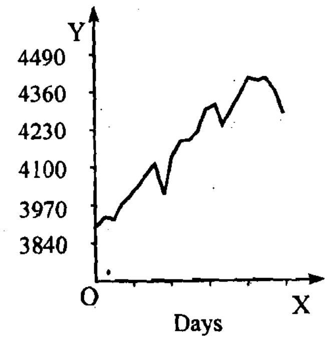


Fig. 1 : A sensex graph over a month

Grace goes on to explain to her students that the basic part of the formula for the derivative is just the formula for the slope of the segment joining two points on the curve. The instantaneous part is where the limit comes in. Taking simple examples, she tells them, "If you want to find the derivative at $x = x_0$, you need to look first at the graph for a clue. Is the curve going up or down? Imagine a tangent to the curve at $x = x_0$ (see Fig. 2). The slope of the tangent line is the slope of the curve at that point. How will you find it numerically?" She draws this tangent, and also the secant joining $P(x_0, f(x_0))$ to a point on the curve near it, say $Q(x_0+h, f(x_0+h))$ (see Fig. 2).

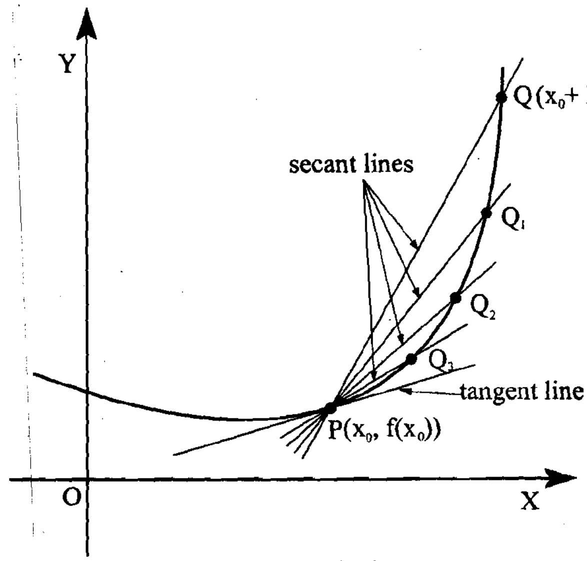


Fig. 2

Then she asks them what happens to this secant as Q moves towards P along the curve. She traces this movement with her finger on the board to a nearer point (Q_1) and nearer to (Q_2), and so on, each time drawing a fresh secant. She points out that the interval $[x_0, x_0+h]$ reduces to smaller and smaller intervals. When they realise this just means that x_0+h is getting closer to x_0, she explains that this is where the limit part comes in. As x_0+h gets closer and closer to x_0, the secant tends to merge with the tangent at $x = x_0$.

Sometimes Grace modifies the explanation a bit by asking the students to take a ruler and keep adjusting it to form secants at nearer and nearer points on the curve. Then they see the secants actually merging with the tangent.

———×———

In the example above the teacher has chosen her way of making the derivative more understandable to students. Do you agree with it? Here's a related exercise.

E1) i) Which situations from your own students' lives could you use to explain the basic idea behind differential calculus to your learners?

ii) How would you introduce your students to 'derivative'?

Let us now consider a common source of confusion for students related to what we have just discussed. This is the relationship between continuity and differentiability.

6.3 CONTINUITY VERSUS DIFFERENTIABILITY

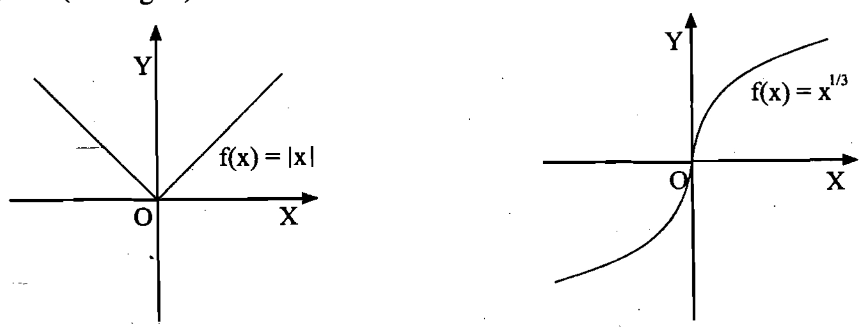
When I ask students of Class 12, or even first-year college students, if every continuous function is differentiable, invariably they say this is true, but not the other way around! This reaction is probably a result of the way we teach and assess them. The students simply mug up a result and its proof without understanding it. As a result, they know that one condition implies another, but which implies which is the problem. Possibly, a good way to help students relate the two conditions is to give several visual examples of functions that :

i) are continuous, but not differentiable at a point;
ii) both continuous and differentiable;
iii) not continuous, **hence** not differentiable.

E2) Give the graphs of at least two functions for each of the three situations listed above. How would you use these examples to clarify your students' understanding regarding the connection between differentiability and continuity?

In workshops I have asked teachers to do the exercise above. Some of them used the method they had worked out in the workshop quite successfully in their classrooms.

They utilised examples like $|x|$ and $x^{1/3}$ to show students situations in which a continuous function would not be differentiable. The thrust of their strategy was to give students examples of a variety of graphs which could be drawn without lifting pencil from paper — some that included sharp corners and some that were smooth throughout (see Fig. 3).



(a) (b)

Fig. 3 : (a) $y = |x|$, (b) $y = x^{1/3}$

They used these examples to show the students that though all the graphs were continuous throughout, the ones with sharp corners (Fig. 3(a)) were not differentiable at the points at which the corners are formed. Also, some smooth curves were not differentiable at the points where the tangents to the curve were vertical (as in Fig. 3(b)). And why is this so?

Why this happens is where the students need to utilise their understanding of derivative as the 'slope of the curve', that is, the limit of the slope of the secant at the point as the secant gets smaller and smaller. Suppose you ask your students to consider the function f given by $f(x) = |x|$ at the sharp corner, i.e., the point $x = 0$. **There is no unique tangent line** to the curve at $x = 0$. If we approach 0 from the left, it appears that the slope of the tangent line should be -1, that is, $y + x = 0$ is the tangent. But if we approach 0 from the right, it appears that the slope of the tangent should be $+1$ that is, $y = x$ is the tangent. So, df/dx doesn't exist at $x = 0$.

Now, let us consider the other situation, shown in Fig. 3(b). Here, the curve has a vertical tangent line at $x = 0$. This means that the limit of the slope does not exist (a case of infinite limits!) at such a point. Therefore, f is not differentiable at $x = 0$.

To emphasise the fact that a function is certainly not differentiable at a point of discontinuity, you could give them several other examples (as suggested in Unit 5). In Fig. 4 below, we show various ways in which a derivative can fail to exist.

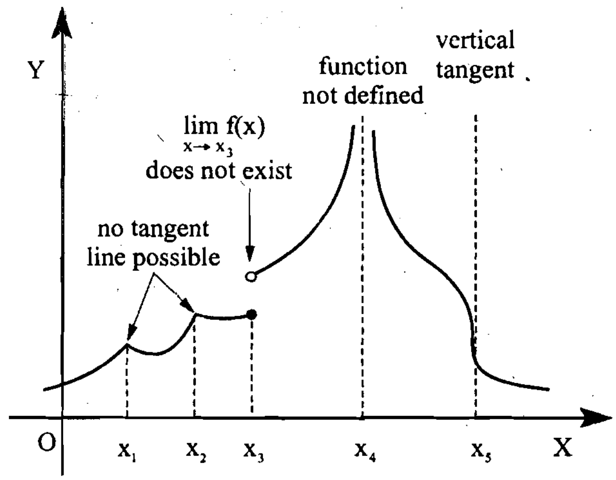


Fig. 4 : The curve is not differentiable at the points $(x_i, f(x_i))$ for i = 1,2,3,4,5.

Here are some related exercises you could do with your learners. In fact, several exercises of this kind could help them become comfortable with the notion of a derivative and its visualisation.

E3) Give your students the following graphs and ask them over which intervals they are :

i) continuous; ii) differentiable.

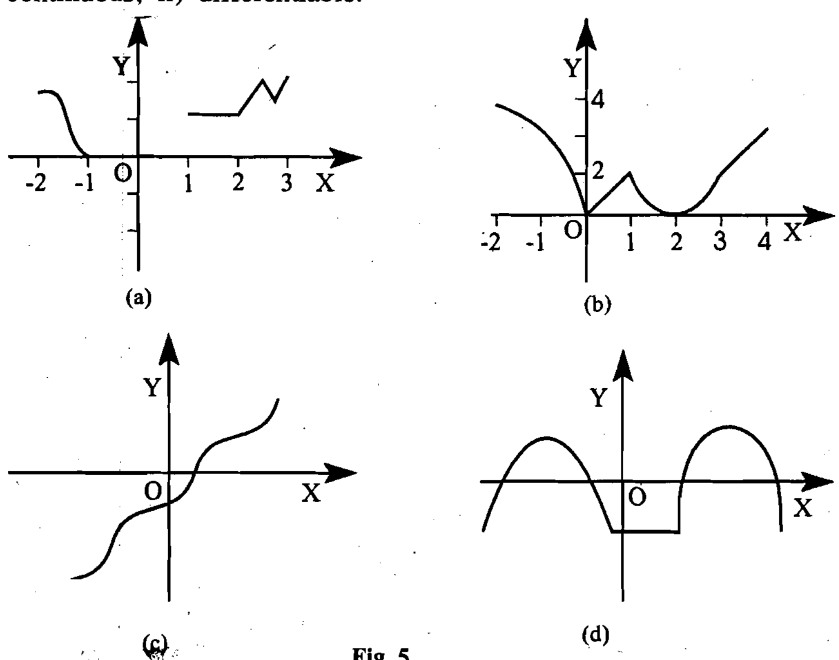


Fig. 5

E4) For each of the following functions, determine whether it is continuous and/or differentiable at x=1. Also, graph these curves and geometrically show what your conclusions mean.

i) $$f(x) = \begin{cases} x+2 & \text{for } -1 \le x \le 1 \\ 3x & \text{for } 1 < x < 5 \end{cases}$$

ii) $$f(x) = \begin{cases} 2x-1 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } 1 < x \end{cases}$$

iii) $f(x) = \begin{cases} x-1 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \\ 2x-2 & \text{for } x > 1 \end{cases}$

iv) $f(x) = (x-1)^{2/3}$

E5) Test each of the functions below for differentiability at the points mentioned alongside. Also graph the corresponding derivative function, if it exists.

i) $f(x) = x^2$ at $x = 0$.

ii) $f(x) = |x - 1|$ at $x = 0$ and $x=1$.

Now let us see how to help our learners see the utility of the derivative of a function for analysing the nature of its graph.

6.4 THE UPS AND DOWNS

As you know, the geometric interpretation of the derivative, and its derivative, are very useful for understanding a function. Helping students to see this aspect, by getting them to graph the functions with the help of the derivative will make this concept more meaningful to the students.

Do your students realise the important role of the first and second derivatives for tracing curves? Ask them : Just by considering f' and f'' at different points, is it possible to trace the graph of f? How? **Examples, and several exercises, to answer these questions is what your learners need to be exposed to.**

The first derivative, f', shows us where the extreme points occur, and where the function f is increasing or decreasing. The second derivative, f'', gives the instantaneous rate of change of the first derivative. So, it tells us **how fast** f is increasing or decreasing. For instance, if s(t) is the distance covered by a vehicle in time t, $v = \frac{ds}{dt}$ is its velocity, and $a = \frac{d^2s}{dt^2}$ is the acceleration of the vehicle, that is, the rate of increase or decrease in the speed of the vehicle.

What is really meaningful in the context of curve tracing is that the behaviour of f'' affects the shape of the graph. Knowing f'', we can find out over which intervals f is concave upwards or downwards. To make this point clear to your students, you could ask them, for instance, to graph the curve of $f(x) = 2x^3 - x^2 - 20x - 10$ on $[-2,4]$. They would need to first look for critical points in $]-2,4[$ (that is, the points x for which $f'(x) = 0$) and the points in $[-2,4]$ where $f'(x)$ does not exist. In this case, the critical points are $x = \frac{-5}{3}, 2$, and $f'(x)$ exists $\forall\, x \in [-2,4]$.

To decide whether the critical points are extreme points, and of what kind, the students would need to find f'' at these points. Since $f''(2) > 0$ and $f''(\frac{-5}{3}) < 0$, 2 and $\frac{-5}{3}$ are a local minimum and maximum, respectively. The students should note that some local extrema can also be absolute extrema, as in the case of (2,f(2)).

The students would also need to find the intervals in which $f'' < 0$ and $f'' > 0$ for deciding the kind of concavity the curve has.

Since $f''(x) = 2(6x-1)$, $f'' > 0$ for $x \in]\frac{1}{6}, 4[$ and $f'' < 0$ for $x \in [-2, \frac{1}{6}[$. So, the curve is concave upwards in $[\frac{1}{6}, 4]$, and concave downwards in $[-2, \frac{1}{6}[$.

The students now need to see, with your help, how the information they have gathered can be used for drawing the curve, which is given in Fig. 6.

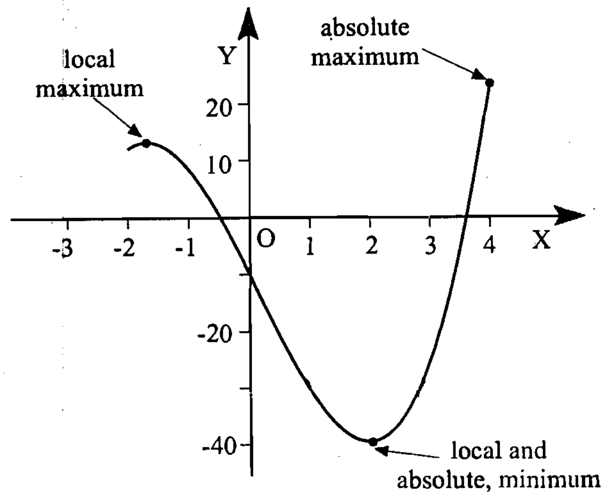


Fig. 6 : The graph of f, defined by $f(x) = 2x^3 - x^2 - 20x - 10$ on $[-2,4]$

We usually give our learners several 'curve tracing' exercises. Here are some exercises that are not usually given. But these would help your learners understand and appreciate this application of the derivative, as well as its use in real-life situations.

E6) i) By looking at the second derivative, decide which of the curves in Fig. 7 could be the graph of $f(x) = x^{5/2}$.

ii) By looking at the first derivative decide which of the curves in Fig. 7 **could not** be the graph of $f(x) = x^3 - 9x^2 + 24x + 1$ for $x \geq 0$.
(**Hint** : Factor the formula for $f'(x)$.)

iii) By looking at the second derivative, decide which of the curves in Fig. 7 could be the graph of $f(x) = \sqrt{x}$.

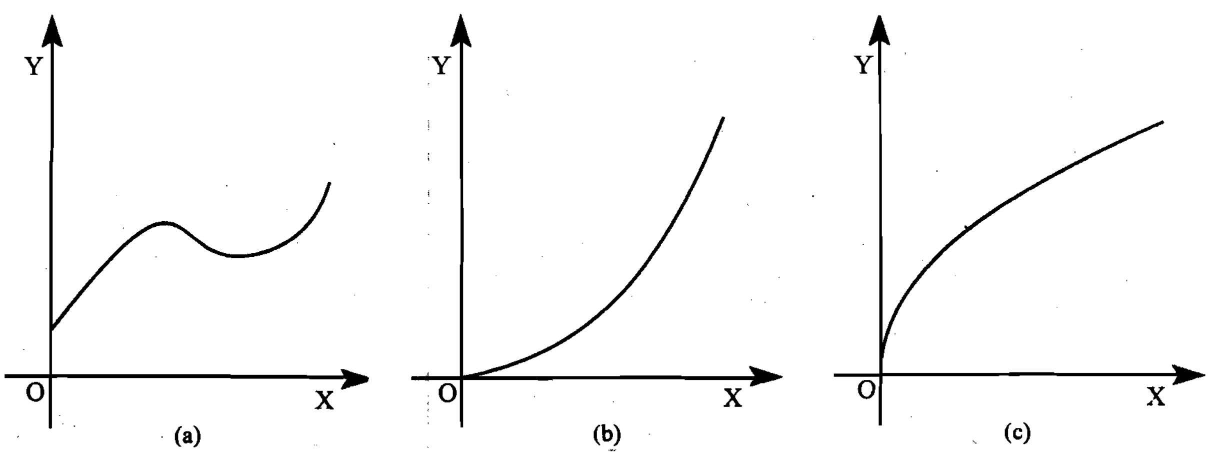


Fig. 7

E7) Sketch the graph of a function f for which :

i) $f(2) = 1$; $f'(2) = 0$; f is concave upwards for all x;

ii) $f(3) = 5$; $f'(x) > 0\ \forall\, x \in]-\infty, 3[$; $f'(3) = 0$; $f'(x) > 0\ \forall\, x \in]3, \infty[$.

iii) $f(3) = -2$, $f'(3) = 2$, $f''(3) = 3$ (sketch in the neighbourhood of x = 3).

E8) An enterprising (although unscrupulous) business student has managed to get his hands on a copy of the out-of-print solutions manual for an applied calculus text. He plans to make photocopies of it and sell them to other students. According to his calculations, he figures that the demand will be between 100 and 1200 copies, and he wants to minimize his average cost of production. After checking the cost of paper, duplicating, and the rental of a small van, he estimates that the cost in rupees to produce x hundred manuals is given by $C(x) = x^2 + 200x + 100$. How many should he produce in order to make the **average cost** per unit as small as possible? What is the least amount he will have to charge to make a profit?

E9) The number of salmon swimming upstream to spawn is approximated by

$$S(x) = -x^3 + 3x^2 + 360x + 5000, \quad 6 \le x \le 20,$$

where x represents the temperature of the water in degrees Celsius. Find the water temperature that produces the maximum number of salmon swimming upstream.

E10) A rope, 4 metres long, is cut into 2 pieces. One piece is shaped into a circle, and the other made into a square. Where should the cut be made in order to make the sum of the areas enclosed minimum?

There are two concepts that show up while tracing curves with using derivatives. These are '**critical** (or **stationary**) point' and 'point of **inflection** (or inflexion)'. Students often confuse them, and/or the relationship between them — Is a point of inflection a critical point? What about vice-versa? The students need clarifications on these questions. What is a good way for helping them in this matter?

One way to explain critical points is to geometrically show that those points at which the tangent to the curve exists and is parallel to the x-axis are critical points. Here, you could ask them to think about why they are called 'critical'. Then you could use several curves that they are familiar with to help them observe and realise why these points are 'critical' — that a curve only attains a local maximum or minimum at a critical point, and/or at points at which the curve is not differentiable.

Also, **ask your students if the converse is true**. To help them think about the converse, you could give them graphs of some functions like $f(x) = x^3$, where $f'(0) = 0$, but $x = 0$ is not an extreme point.

Regarding 'point of inflection', ask your students to consider curves like those in Fig. 8.

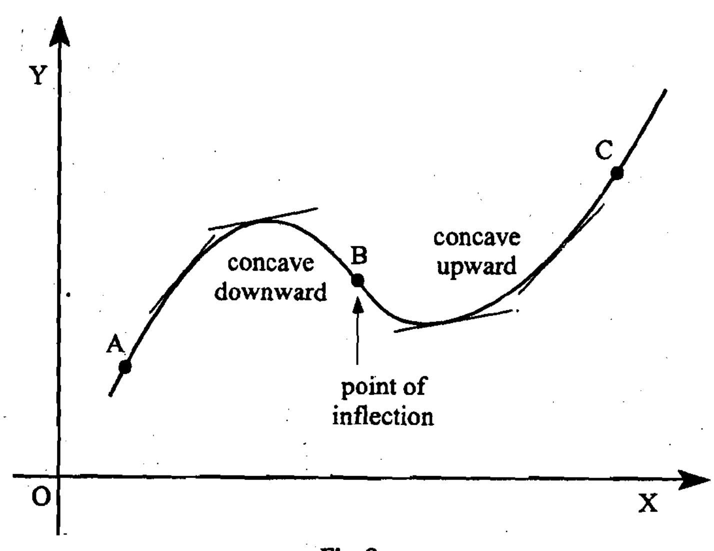


Fig. 8

Ask your students if they notice any difference in the position of the curve relative to the tangents to the curve at any point between A and B, vis-à-vis the position relative to the tangents drawn at any point between B and C. The function lies **below** the tangents from A to B, and **above** the tangents from B to C. So, the curve is **concave downwards** from A to B and **concave upwards** from B to C. Because of this, B is a point of inflection for this curve. More examples and non-examples of such points can be given to students to explore. This will help them develop their understanding of the concept.

Why don't you try an exercise now?

E11) What method would you use for explaining 'critical point' and 'point of inflection' to your learners? Try it with them. How did you judge the effectiveness of your strategy?

In the strategy you have just suggested, did you expose your learners to any real-life examples of points of inflection? We have suggested some such examples below.

Point of Diminishing Returns : Let us start with considering Fig. 9.

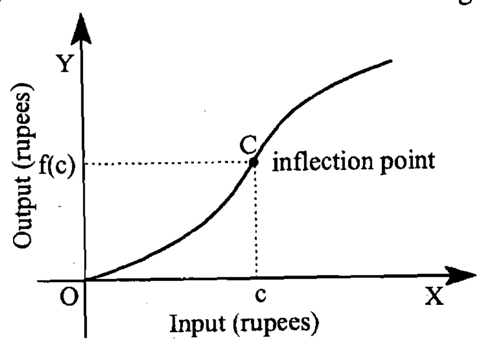


Fig. 9

Extreme values of the slope always occur at inflection points.

The graph in this figure depicts the output of a factory worker over a period of time. To start with, the graph is not very steep. The steepness increases further on, until the graph reaches a point of maximum steepness (at C) after which the steepness begins to decrease. This tells us that at first, the worker's rate of production is low. The rate of production increases as the worker settles into a routine, and continues to increase until the worker is performing at maximum efficiency. Beyond this point fatigue sets in, and the rate of production begins to show a decline. The point of maximum efficiency is known in economics as the **point of diminishing returns.**

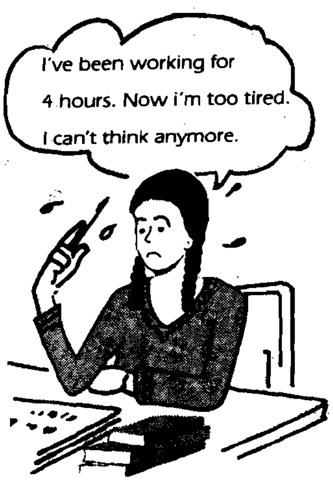


Fig. 10

The behaviour of the graph on either side of this point C can be described in terms of the slope. To the left of this point the slope of the tangent increases as t increases. (This indicates that the output is increasing at a faster rate with each additional hour spent by the worker.) To the right of the point C, the slope of the tangent decreases as t increases. (This indicates that the increase in output is smaller with each additional hour spent on the job.) It is this increase and decrease of the slope on either side of this point that tells us that C is a point of inflection of this function. In this situation it shows us that C is the point of maximum efficiency, that is, the point of diminishing returns. Any input beyond this point of time corresponding to C will not be considered to be a good use of this worker's labour.

Point of Maximum Efficiency : Ask your students to think of their own situation when preparing for an exam. A couple of days before the exam they sit down to revise their syllabus. At first, they will be studying slowly. As they set their mind to do more, efficiency increases, and it increases up to a maximum level, say, 4 hours after beginning to study. After that point of time, the ability to concentrate slowly declines, and many say, "my mind is fully drained, and I can't think any more". This point, which is the point of maximum efficiency (the point of diminishing returns), is a point of inflection in this situation.

Point of Maximum Yield : The graph in Fig. 11 shows the population of catfish in a commercial catfish farm as a function of time. As the graph shows, the population increases rapidly up to a point, and then increases at a slower rate. The horizontal dashed line shows that the population will approach some upper limit determined by the capacity of the farm. The point at which the rate of population growth starts to decrease is the point of inflection for the graph.

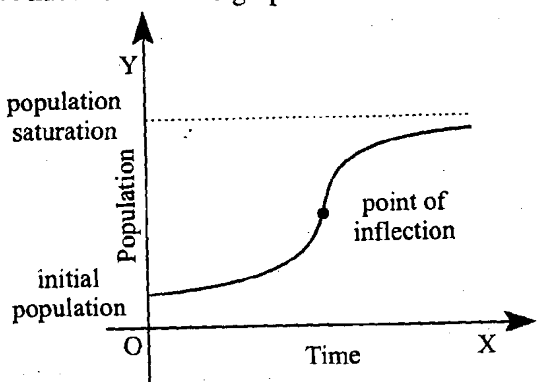


Fig. 11

To produce the maximum yield of catfish, harvesting should take place at the point of fastest possible growth of the population. This is at the point of inflection. The rate of change of the population, given by the first derivative, is increasing up to the inflection point (on the interval where the second derivative is positive) and decreasing past the inflection point (on the interval where the second derivative is negative).

Why don't you try the following exercise now?

E12) List two real-life problem situations that you can give your learners to enable them to understand 'critical point' and 'point of inflection'.

Let us now consider another question that students frequently wonder about regarding the shape of the graph of a function. The question is:

Is there a relationship between concavity and the type of monotonicity of a curve? Through examples of the kind given in Fig. 12, you could help your students to note that a function can be either increasing or decreasing on an interval regardless of whether it is concave upwards or concave downwards on the interval.

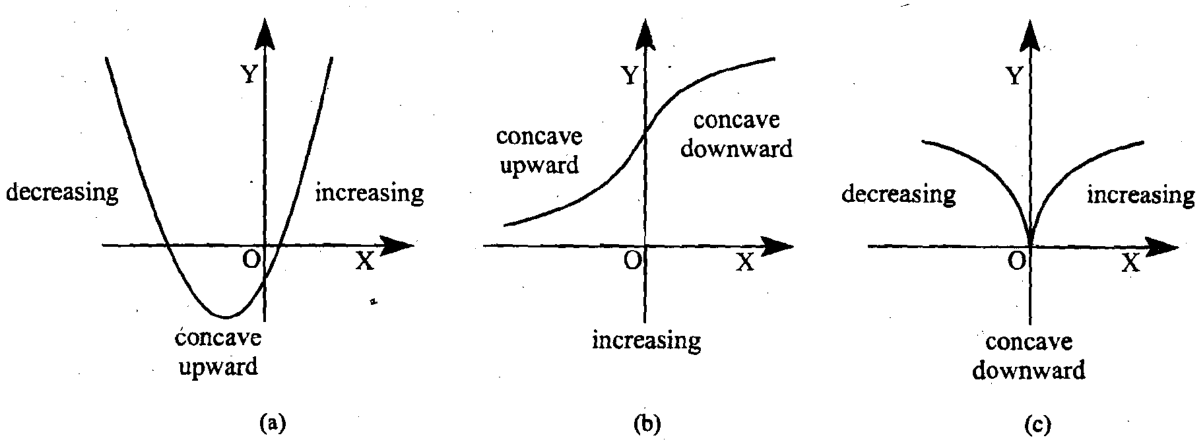


Fig. 12 : A function that is (a) concave upwards in [a,b]; (b) concave upwards in [a,0], and concave downwards in [0,b]; (c) concave downwards in [a,b].

The following chart shows how a graph may combine the properties of increasing, decreasing, concave upwards and concave downwards.

Sign of f'	Sign of f''	Increase or Decrease in f	Concavity (C) of f
+	+	Increasing	C-up
+	–	Increasing	C-down
–	+	Decreasing	C-up
–	–	Decreasing	C-down

You should also ask your students to think of examples of each category in the chart above.

Now, your students would have learnt that at a point of inflection of a function f, either $f''=0$ or f'' does not exist. But, have you ever asked them **if the converse is true**? That is, if $f''(x_0)=0$, is x_0 a point of inflection of f? You could give your students examples like $y = x^4$ that can help them answer the question themselves. Here, $f''(0)=0$, but $f'' > 0$ when $x > 0$ and when $x < 0$. So there is no change in concavity at $x = 0$.

You also need to expose your learners to situations in which $f''(x_0)$ does not exist, but x_0 is a point of inflection for f. Think about these aspects while doing the following exercises.

E13) Give examples of functions f, defined on an interval [a,b], such that

i) f' and f'' exist in [a,b];
ii) f' exists in [a,b], but f'' does not exist at some points of [a,b];
iii) f' does not exist on [a,b].

Also find the critical points of these functions, as well as points of inflection, if any.

E14) Give two examples, with justification, of functions f for which
i) $f''(a) = 0$ but a is not a point of inflection of f
ii) f is concave upwards on some interval, concave downwards on another interval, and yet f has no point of inflection.

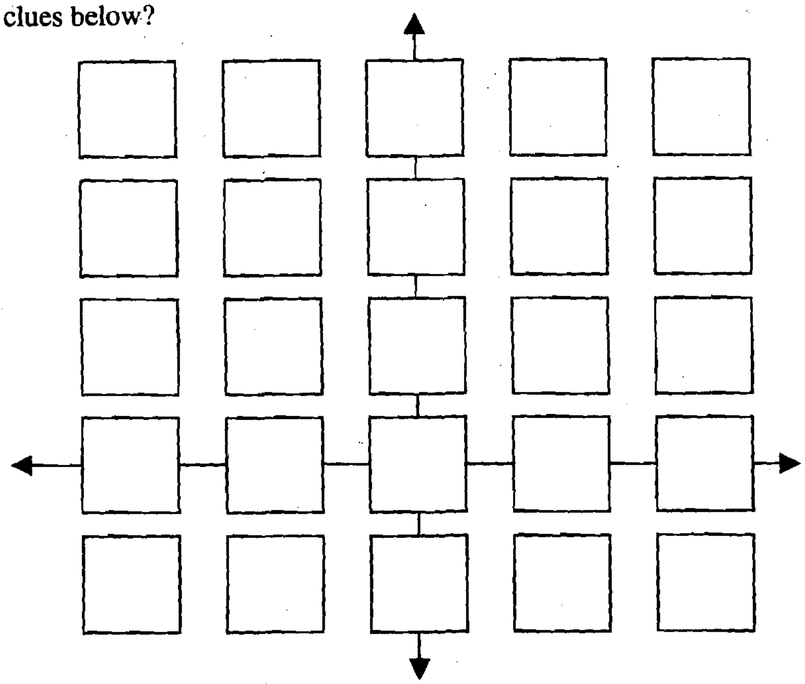
Let us sum up the essence of what we have just discussed in the following remark.

Remark : The first order critical points decide about the extremum of a function whereas the second order critical points determine the change in concavity of f. This shows that critical points of the first order tell us the quantitative nature of a function, and second-order critical points decide the shape of the graph (qualitative behaviour of the function f). The points of inflection of f are the extreme points of f'. At a point of inflection, there is a change in concavity of the curve. But change in concavity itself will not lead to a point of inflection.

Before ending the section, let me suggest an interest activity for your students that cover all the aspects of what you discuss with them in differential calculus. This also exposes the students to the names of several mathematicians.

Activity (Class, Take Your Seats)

Can you fill in the first initial of each student in this math teacher's seating chart using only the clues below?



CLUES:

1. All students are located at integral coordinates in the xy-plane. The x-coordinates belong to the set $\{-2, -1, 0, 1, 2\}$, and the y-coordinates belong to the set $\{-1, 0, 1, 2, 3\}$.

2. Wallis is seated on the line which is normal to the curve $f(x) = x^2 - 2x + 4$ at its minimum point.

3. Newton is seated at a point of inflection of $f(x) = 4x^2 + \dfrac{32}{x}$.

4. Euler sits at the point on the curve $2y = (x - 2)^2$ which is nearest to Boole.

5. MacLaurin is located at the relative maximum point of the function $f(x) = x^3 - 3x^2 - 9x - 4$.

6. Saccheri is seated at the absolute maximum point of the function $f(x) = -x^2 + 4x - 1$.

7. Riemann's seat is one of the critical points of the curve $f(x) = \dfrac{x^4}{4} - x^3 + x^2 - 1$.

8. The function $f(x) = x^2 + \dfrac{k}{x}$ has a point of inflection at $x = 1$. Zeno sits at this point.

9. Boole is seated at the absolute maximum point on the curve $(x - 2)^2 + y^2 = 1$.

10. Archimedes is located at one of the vertices of the rectangle with the largest area that can be drawn with its upper vertices on the line $y = 1$ and its lower vertices on the parabola $y = x^2 - 2$.

11. Thales sits at a point on the curve $f(x) = 2x^3 - 6x^2 + 43$ where the slope is 48.

12. Leibniz sits at a point on the curve $y = \cos(x)$ where the 99^{th} derivative of that curve is 0.

13. Kronecker sits on the line which is tangent to the curve $y = 4x^2 - 22x + 35$ at the point (3, 5).

14. Fermat is seated at the point of inflection of the curve $y = x^3 - 6x^2 + 33x - 51$.

15. Descartes is located at one of the critical points of the curve $y = -3x^4 + 6x^2$.

16. Cantor is located on the line tangent to the curve $y = -x^2 + 10x - 25$ at its maximum point.

17. Gauss sits at the absolute maximum point on the curve $4y = -2x^3 + 3x^2 + 7$ over the interval $[-1, 2]$.

18. Viete's seat is collinear with those of Gauss and Kronecker.

19. Heron is located at the point of inflection of the curve $f(x) = x^3 - 3x^2 + 3x + 1$.

20. Pascal lies on the line tangent to the curve $12y = 16 - 6x^2 - x^3$ at its point of inflection.

(This activity is designed by David Pleacher, 1991 VCTM Mathematics Teacher.)

The activity above can also be done by the class as a whole, divided into teams.

Let us now summarise what we have covered in this unit.

6.5 SUMMARY

In this unit we have discussed the following points.

1. We have given some suggestions for relating the derivative to the students' real-life experiences.

2. Examples have been given for clearing the confusion students have regarding the relationship between continuity and differentiability. In particular, we have spelt out various situations in which a function is not differentiable at a point.

3. You have studied examples to help students realise the significance of f' and f'' for understanding the behaviour of f and the shape of its graph.

4. We have particularly focussed on critical points and points of inflection, geometrically, algebraically and through real-life examples.

5. Stress has been laid on encouraging students to think about whether any critical point is an extremum, and whether $f''(x_0) = 0$ means that x_0 is a point of inflection.

6. We have also suggested that you discuss the fact with your students that there is no relationship between monotonicity and type of concavity of a function.

6.6 COMMENTS ON EXERCISES

E1) i) The examples would necessarily deal with movement of some kind. They could relate to profit or loss, movement of water in a stream, movement of vehicles around them, students running/walking/swimming, etc.

ii) You may find it interesting to note down in **your logbook** any changes in method of teaching the derivative that you have made now. Also, note down the consequences for the quality of learning.

E2) Your students must be familiar with several examples. But did they realise that these examples (like $|x|$) would be appropriate?

While writing your teaching strategy, note down the methods you used for assessing the usefulness of the strategy from the learning point of view.

E3) For instance, did they realise that the graph in (a) is continuous over $]-2,-1]\cup[1,3]$, and differentiable over $]-2,-1]\cup[1,3]\setminus\{2,2.5,2.75\}$?

E4) i) Continuous, not differentiable at $x = 1$; sharp corner at $x = 1$.

ii) As in (i) above.

iii) Discontinuous, and hence not differentiable, at $x = 1$.

iv) Continuous, but not differentiable at $x = 1$; tangent is vertical at $x = 1$.

E5) i) Differentiable at $x = 0$. The curve of f' is the straight line $y = 2x$.

ii) Differentiable at $x = 0$, but not at $x = 1$. The curve of f' over $\mathbf{R}\setminus\{1\}$ is the union of two half lines.

E6) i) Since $f'' > 0\ \forall\, x > 0$, f is concave upwards in $[0,\infty[$. Therefore, of the graphs given, the one in (b) is the most appropriate.

ii) Since $f'(x) = 3(x-4)(x-2)$, f is increasing in $[4,\infty[$ and $]-\infty, 2]$, and decreasing in $[2,4]$. Hence, (a) could be the graph of f, not (b) or (c).

E7)

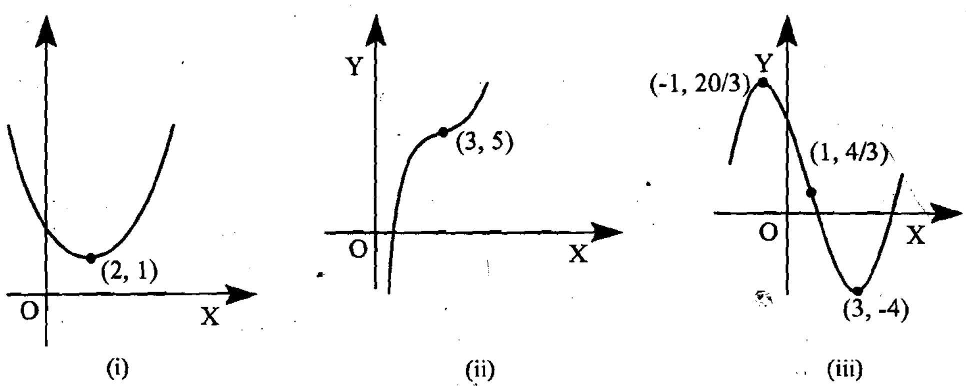


Fig. 13

E8) The average cost function is $\frac{C(x)}{x}$. This is minimum for $x = 10$. Remember, the unit of x is hundred manuals. So, to minimise the average cost, 10 hundred, i.e., 1000 manuals at least should be produced.

The cost of producing these is $\frac{C(10)}{1000}$ = Rs.2.20. So, he should charge more than Rs. 2.20 to make a profit.

Remember to check the endpoints also when looking for absolute minima or maxima.

E9) 12° Celsius.
(Here note that students often forget to write the unit. You need to insist on the units being written.)

E10) Suppose a portion of length x cm. is formed into a circle and the length $400 - x$ cm. is made into a square.

The sum of the areas is $\pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{400-x}{4}\right)^2$.

This is minimum for $x = \left(\frac{400\pi}{4+\pi}\right)$cm.

E11) Here, you should try out methods based on the core course 'Teaching-Learning Process and Evaluation' that you have studied.

E12) For instance, a small company that makes and sells bicycles determines that the cost and price functions for $x\ (\geq 0)$ cycles per week are $C(x) = 100 + 30x$ and $P(x) = 90 - x$, respectively. What are the possible values of x for which the cost would be minimised?

Here, the question only requires the critical points to be determined. However, it can be altered for finding maximum revenue, etc.

Similarly, consider the problem of drug concentration : The percent of concentration of a certain drug in the bloodstream x hours after the drug is given is $K(x) = \frac{3x}{x^2+4}$. Find the time at which concentration is a maximum.

There are several other problems that you can think of related to chemical reactions, velocity and acceleration, etc.

E13) i) Any polynomial function, for example.

ii) For instance, $y = x^{3/2}$ in $[-1,1]$.

iii) For instance, $f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & \text{otherwise} \end{cases}$ on any interval.

Just by looking at the type of functions, ask your students if they can tell whether critical points or points of inflection exist.

E14) i) For instance, take $f(x) = x^4$ over $[-1,1]$, Here $f''(0) = 0$, but $f''(h) > 0$ for $h < 0$ and $h > 0$. So $x = 0$ is not a point of inflection.

ii) For instance, $y = \frac{1}{x}$ is concave downwards at $x = -1$ and concave upwards at $x = 1$. But there is no inflection point in between.