UNIT 4   SETS AND FUNCTIONS

4.1   INTRODUCTION

One of the chief abilities that a student of mathematics needs to develop is the ability to look for patterns and relationships, and to describe and generalise them. One concept that is necessary for their ability to describe relationships of change between variables, explain parameter changes and interpret and analyse the graphs of such relationships is that of a 'function'. And, to describe and understand a function, the student must be clear about the concept of a 'set'. So, the concepts of 'set' and 'function' are basic to mathematics. Therefore, the students must understand them well.

"'Sets' are not a problem. We know that chapter very well", said a Class 11 child to me. The teachers we met also told us the same. However, they pointed out that certain functions, and finding certain sets related to them, were not clear to the students. Through this unit, we aim to help improve this situation. We suggest some teaching strategies, exercises, activities and projects that may help the students in improving their understanding.

In the unit, we start with supplementing the material on 'Sets' given in the NCERT textbook. Then we pick out certain functions that are connected with real world situations. We suggest that such functions should be used as a base for introducing the children to the concept of a function, its domain and range, and other properties of functions. And, finally, we spend quite some time on ways of helping children use relationships between functions to draw their graphs.

Objectives

After studying this unit, you should be able to develop the ability of your learners to

- give examples and non-examples of sets, with justification;
- relate the geometric and algebraic representations of Cartesian products;
- explain the difference between a function and a relation;
- explicitly present the domain and range of functions that they come across;
- correlate the algebraic and geometric representations of a function;
- construct the graphs of f±g, fg, f ± c, |f| if the graphs of the functions f and g are given, where c is a constant;
- define the inverse of a bijective function, and show the graphs of such a function and its inverse.

4.2 SOME THOUGHTS ON SETS

Usually, when introducing a learner to the concept of a set, we give them examples of the various number systems and their subsets. As non-examples, we give collections of objects which are not well-defined. This concept of 'well-defined', though, remains very unclear to the students. When talking to them, one of the common misconceptions I have found is that a set is any collection of numbers (or letters) only. So, for example, for such a student the collection of Navodaya Vidyalaya mathematics teachers is not a set.

Why don't you try some exercises now?

E1) In your discussions with your learners on 'Sets', which other learning difficulties and misconceptions have you found?

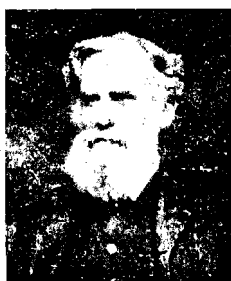E2) Is {N, India, $\sqrt{2}$, IGNOU} an example of a set? Give reasons for your answer.

You must have tried E2. Did you also ask your students to try it? What were their answers? Did most of them say it wasn't a set? This is not surprising because such examples of sets are very rarely presented to the students. They need to be helped to realise that **this is an example of a set** presented in roster form with its 4 elements clearly listed. The criterion for an element to belong to this set is that it should be one of these four only. So, this is a well-defined collection.

Fig. 1 : John Venn
(1834-1923)

Now let us take a quick look at some pictorial representations of sets, that is, Venn diagrams. (Here, a note on who Venn is may interest your learners.) Students often use them to 'prove' that, for example, $(A \cap B)' = A' \cup B'$, where A and B are two sets and A' denotes the complement of A. In fact, several children have given me 'proofs' like the one in Fig. 2.

When I ask these students how they can say this argument is true for **any two** sets, they get confused. They don't realise that just showing one situation does not constitute a proof — we have to check the statement for all possible cases like $A \subseteq B$, $A \cap B = \varphi$, etc. Alternatively, we could show them how the algebraic way of proving such statements allows us to prove the general case at one go. An algebraic proof, of course, also helps them learn about precision, the use of deductive logic, and using symbols correctly.

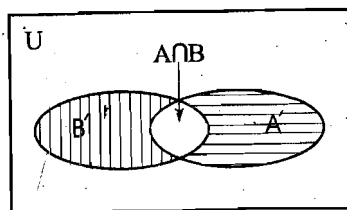You could try the following exercise with them now.

Fig. 2

E3) Give your students a real-life situation involving three sets, and ask them to check some property about these sets. Now, ask them to check this property for all possible sets. Note down the way the children tried it. What kind of errors did they make? How did you help them rectify their misunderstandings?

Venn diagrams are a schematic method of representing various sets. But there are certain sets that have conventional geometric representations that we are familiar with. For instance, **R** is represented by the number line, $\mathbf{R}^2$ by the plane, etc. Interestingly, very few students realise that $\mathbf{R}^2$ is the Cartesian product of **R** with itself. This is because most teachers introduce their students to **the Cartesian product** of two sets A and B by defining it formally as $A \times B = \{(a,b) \mid a \in A, b \in B\}$, followed by some formal examples and exercises. Very rarely are the students given real-life situations

that are modelled by Cartesian products. What's worse, very few teachers bother to provoke their students to think about the geometric meaning of A × B.

For instance, what is the geometric view of {1} × {3}? Isn't it just the point (1, 3) in $\mathbf{R}^2$? And, what about $\{-1,1\}\times\{2,5\}$? When I asked students this, many of them showed me the representation given in Fig.3(a). Some showed me the representation given in Fig.3(b). Very few, only two in fact, showed me the correct representation shown in Fig.3(c).
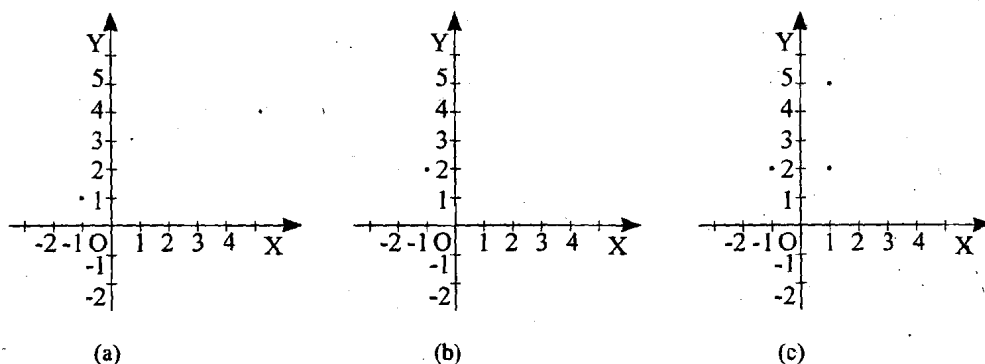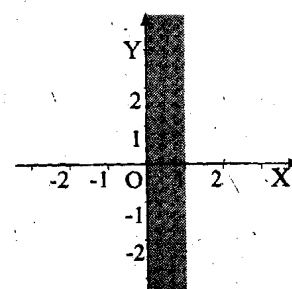

(a)　　　　　　(b)　　　　　　(c)

Fig.3

What is the geometric representation of [0, 1] × **R**? The students shouiu be allowed to take their time to understand why it is the infinite shaded area shown in Fig. 4, namely, $\{(x,y) \mid x \in [0,1], y \in \mathbf{R}\}$.

You could slowly build up their understanding by asking them to give the geometric representations of, say, {1} × **R** and **R** × {1}. These are the lines x = 1 and y = 1, respectively, in the xy-plane. Such exercises will also help them realise that A × B ≠ B × A, that is, the Cartesian product is not commutative. (Is it associative? Check!)

And, what is S^1 × [0, 1], where S^1 is the unit circle? Which of the figures in Fig. 5 represent it?

Fig. 4 : [0,1] × R
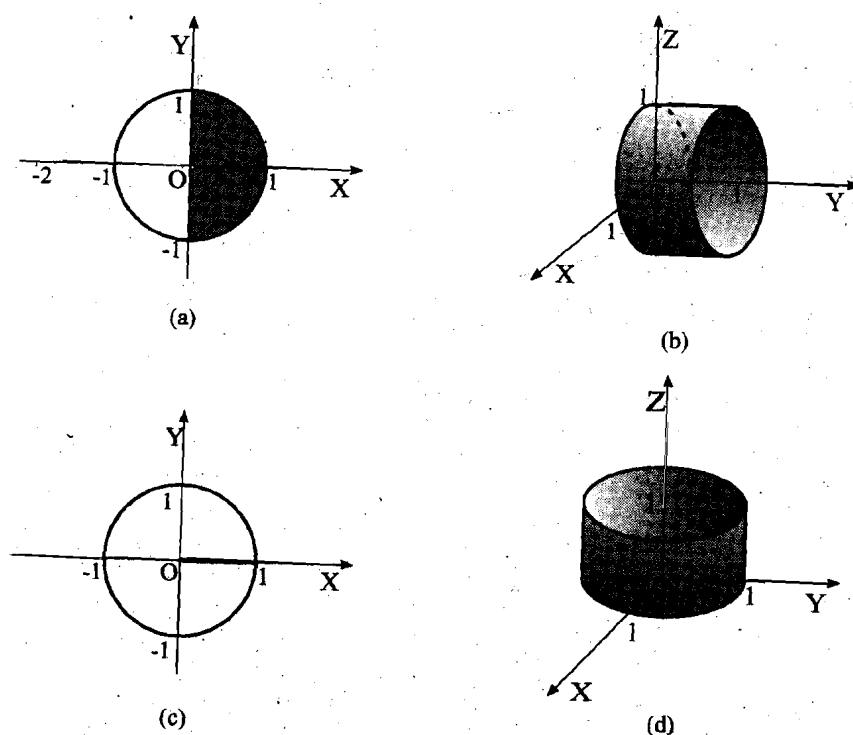

(a)

(b)

(c)

(d)

Fig. 5

These are the kinds of questions that can help children see this connection between algebra and geometry. Think about this while trying the following exercises with your learners.

E4) Give three situations from the lives of your learners the mathematical models of which require the use of Cartesian products of sets.

E5) Give the geometric representations of

i) $[2, 3] \times [2, 3]$,

ii) $[0, 1] \times \{2\} \times [0, 1]$, and

iii) $([2, 3] \cup [-1, 1]) \times [-1, 1]$.

E6) While trying E4 and E5, what types of difficulties did your students face? What kinds of errors did they make?

Let us now discuss some aspects of a related and very basic concept of mathematics about which quite a few students have conceptual problems.

4.3 FUNCTIONS GALORE!

To begin with, why don't you recall the strategy you use for introducing a function to children? Do you give them examples from their surroundings and/or from other subjects they're studying? The following exercise is about this.

E7) List 5 situations related to your learners' daily lives that give rise to functions. Clearly define the functions in each situation, and their domains and ranges.
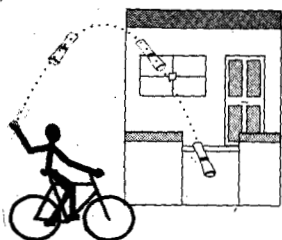

Fig.6 : The path of a projectile is parabolic.

Regarding the exercise above, one teacher gave me an interesting answer to it. She said, "There is an example which works beautifully. Consider the rule which associates every man with his wife. This would be a function from the set of all men into the set of all women **provided** that every man was married and polygamy was prohibited. In this case, bachelors would not exist, the range would be the set of all wives (i.e., married women). Some women could remain single (unless the function is made onto). Polyandry would be legal (unless the function was one-to-one)!

Depending on the sensibilities of the class, one could tell it like a story with embellishments: God created men, who got lonely. So God created women (without bothering to count), and asked each man to choose, a wife, but only one, so as not to be greedy! Several men chose the same woman, and some women remained single. Later on the wives' association took a delegation to God and said, "One is bad enough! Why do we have to cope with more than one?" So God made the function one-to-one, and so on.

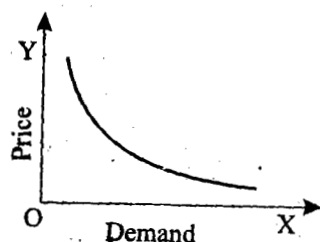For gender sensitivity purposes, the roles of men and women can be interchanged off and on!"

Some people like to introduce their students to functions through graphs that the children may come across in their environment — on TV, in magazines, while studying other subjects, and so on. For instance, while studying projectiles, they see graphs like the one in Fig.6. While studying economics, they see demand and supply curves like the one in Fig.7. In newspapers and magazines, they may see temperature curves as in Fig.8.

Fig. 7 : Demand curve

Using these pictorial representations, you can ask various questions that would nudge them towards arriving at the defining properties of a function. For instance, you could ask them to observe how many values of y (the dependent variable) correspond to a given value of x (the independent variable), if there is a value of y corresponding to every value of x, and so on. So, for example, for each demand point, we get a price corresponding to it (in Fig.7), and only one price.

You could also use such graphs to clearly demonstrate the domain and range of the function. For instance, in Fig.7, you can help them see that the domain is $\{x \in \mathbf{R} \mid x > 0\}$ and the range is $\{y \in \mathbf{R} \mid 0 < y \le 10\}$. In fact, children can be helped to understand many aspects of functions in several ways. A glimpse into one such classroom interaction is given below.

Example 1 : A government school teacher, Ms.Kamla, had invited me over to observe the teaching/learning processes in her classroom. When I arrived, she had already introduced her students to functions and some related concepts. She began the present class with asking them a few review questions. Then she asked her students to do the following problem :

Give me a function f which has all the following properties :

i) f is undefined at x = –3

ii) f has a zero at x = 2

iii) the domain of f is $[-6, \infty[\setminus \{-3\}$

iv) f(x) ≥ 0 ∀ x in the domain of f

v) (5, 15) lies on the graph of f.

Fig. 8 : Temperature curve

Kamla allowed the children to discuss the problem in groups. While their discussions were going on, she went around the groups, sitting with them and sharing their ideas. After 15 minutes, the groups were asked to present whatever ideas they had gathered till then.

Group A started with giving $f(x) = \dfrac{x-2}{x+3}$. Kamla asked the presenter to explain the reason for this. He explained that the first property required f to have x+3 in the denominator, and the second property was satisfied by x – 2 in the numerator. Here, Kamla asked him how the other properties were satisfied. At this point many children started shouting out various answers. So, she asked Group B to come up and give their response.

This presenter modified the earlier function to $f(x) = \left|\dfrac{x-2}{x+3}\right|(\sqrt{x+6})$. But, some other students objected, saying that now property (v) was not satisfied because $f(5) = \dfrac{3}{8}\sqrt{11} \ne 15$. "So", asked Kamla, "how can we change f to get f(5) = 15?" Several suggestions came forth, namely, add $\left(\dfrac{3}{8}\sqrt{11} - 15\right)$, multiply by $\left(\dfrac{3}{8}\sqrt{11} - 15\right)$, etc.

Each suggestion was tried out, modified or rejected. With more such discussions, the final definition that the class agreed to was

$$f(x) = \dfrac{40}{\sqrt{11}}\left|\dfrac{x-2}{x+3}\right|\sqrt{x+6} \text{ for } x \in [-6, \infty[\setminus \{-3\}.$$

However, two children in Group D felt that they had another solution. Kamla asked them to present it also. According to them, since y = 5(x – 2) is the line passing
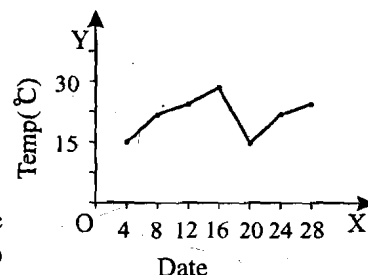
through (0,2) and (5,15), $f(x) = \sqrt{x+6}\left|\dfrac{5(x-2)}{x+3}\right|$ is also a solution. But some children said that, again, (5, 15) did not lie on this. So, these two modified it to $f(x) = \sqrt{x+6}\left|\dfrac{5(x-2)}{x+3}\right|\dfrac{15}{f(5)}$. Ultimately, they got the same function.

By now the class was ending. So, as homework, Kamla asked all the children to see if other functions could also be found with the same properties.

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Apart from interactions like the one above, real-life problems can be brought into the classroom for the children to try and analyse and solve. For instance, you may be interested in the following report from a teacher, Chitra.

Example 2 : Chitra had given several examples of functions in daily life to her Class 11 children, and asked them to think of many more. In this class she asked them to consider the following problem :

A tempo carrying many bags of rice to the district godown starts out along a straight road at 50 km.p.h. After a half hour, the driver of a passing car tells the tempo driver that one bag fell off the tempo a few kilometres earlier. The tempo driver retraces his route at a speed of 30 km.p.h. After 10 minutes he finds the bag. He takes 5 minutes to load it back on and tie the bags carefully. Then he rushes to the godown at 70 km.p.h. Three quarters of an hour later he has reached the godown.

In the situation above, let s be the function describing the distance of the truck from its starting point at time t. Give an algebraic rule for defining s, and sketch its graph.

There was a lot of discussion among the children. All the students wanted to first draw up a table of values. So, this was finally done on the board with all the children giving their input.

t(in min.)	0	30	40	45	90
s(t)	0	25	20	20	72.5

Here Chitra pointed out that this was only a table giving values at the times mentioned. Weren't other values needed for drawing the graph? Immediately, a student, Mahmooda, said that she could already draw the graph, which she did on the board (as shown in Fig.9).
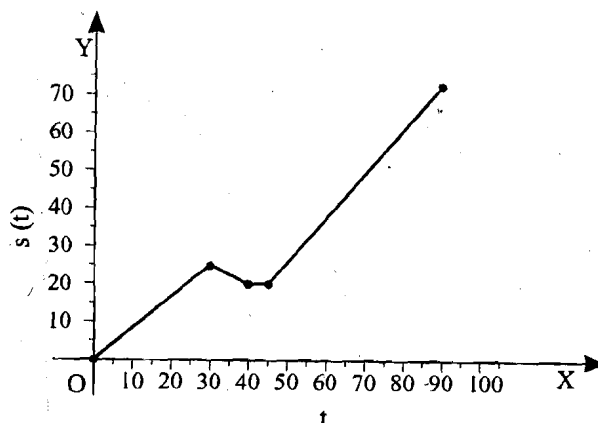

Fig. 9

Some children expressed their doubts about this curve because they said she had only joined the points in the table. How did she know that the curve would behave like this in between also?

At this point, the teacher asked all the children to divide up into groups. Then she asked all the groups to have a discussion and come up with the algebraic definition of the function. Half an hour later, and with a lot more of 'buts and hows' and trying out and rejecting various rules, the children jointly came up with the following definition.

$$s(t) = \begin{cases} 50t & , 0 \leq t \leq \frac{30}{60} \\ 25 - 30t, & 0 \leq t \leq \frac{10}{60} \\ 20 & , \frac{40}{60} \leq t \leq \frac{45}{60} \\ 20 + 70t, & 0 \leq t \leq \frac{45}{60} \end{cases}$$

Now, Chitra asked the children to carefully check what was written on the board. She wondered, for instance, how there can be two different rules in the interval $0 \leq t \leq \frac{10}{60}$?

Are there two different functions, she asked? As expected, this led to a lot of confusion. The children tried to sort out the problem in different ways, but to no avail. Finally, Chitra decided to teach them how to translate the interval. She told them that since the second interval should be from $\frac{30}{60}$, i.e., $\frac{1}{2}$ onwards, she would add $\frac{1}{2}$ to this part of the interval so it would become $\frac{1}{2} \leq t + \frac{1}{2} \leq \frac{10}{60} + \frac{1}{2}$ $\left(= \frac{2}{3}\right)$.

Now, call $t + \frac{1}{2} = t'$, then $t = t' - \frac{1}{2}$ and

$$s(t') = 25 - 30(t' - \frac{1}{2}) \text{ for } \frac{1}{2} \leq t' \leq \frac{2}{3}$$

$$= 40 - 30t', \text{ for } \frac{1}{2} \leq t' \leq \frac{2}{3}.$$

And then she reminded them that whether we write $s(x) = 40 - 30x$, $\frac{1}{2} \leq x \leq \frac{2}{3}$, or $s(t) = 40 - 30t$ for $1 \leq t \leq \frac{2}{3}$, or $s(t') = 40 - 30t'$, $\frac{1}{2} \leq t' \leq \frac{2}{3}$, the rule of definition remains the same. So we can write $s(t) = 40 - 30t, \frac{1}{2} \leq t \leq \frac{2}{3}$.

Again, for the last piece of the definition of s, she asked the children to re-define it from $t = \frac{45}{60}$ on. They did it similarly, and finally defined s throughout as below

$$s(t) = \begin{cases} 50t & , 0 \leq t \leq \frac{30}{60} \\ 40 - 30t, & \frac{30}{60} \leq t \leq \frac{40}{60} \\ 20 & , \frac{40}{60} \leq t \leq \frac{45}{60} \\ 70t - 32.5, & \frac{45}{60} \leq t \leq \frac{90}{60} \end{cases}$$

Next, she got them to relate their definition with the graph Mahmooda had drawn and see if they tallied. When the students did so, they readily accepted her geometric representation of s(t).

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Chitra just reported one such interaction. Two sources of errors that were clarified during this discussion, but haven't been mentioned above, are commonly found in children. They are given below.

- Most of the children believe that the units taken along both the axes have to be the same. For instance, Mahmooda and others, in the example above, couldn't understand that if 10 minutes are depicted by a 1 cm. length along the t-axis, then 10 km. can be depicted along the s-axis by a 2 cm. length.

- Most children tend to ignore the units of measurement used or required in the problem. For instance, in the example above the time interval had to be measured in hours if distance was to correspond to km. per hour. Otherwise, the distance values would need to be altered to those corresponding to km./min.

Such misunderstandings, and others, can be cleared by allowing the children many opportunities to apply their minds on these points. Each time we must get them to articulate the reason for choosing the same or different unit lengths (for instance).

Many exercises need to be done to build the conceptual understanding of the learners, and, in particular, to help them relate the algebraic and graphical representations of functions. You may like to do the following ones yourself, and with your learners.

E8) Give an example, with justification, of a curve which does not represent a function.

E9) Give the algebraic and graphical representations of a function whose domain is $\{x \in \mathbf{R} \mid x < 0\} \cup \{x \in \mathbf{R} \mid x > 5\}$ and range is $\{1, 2, 3, 4\}$.

E10) How would you help a.student realise that two different looking graphs can actually represent the same function? (e.g., if you use different unit lenths along either axis, the graph changes.)

One of the difficulties that learners face in the context of functions is related to the composition of functions — when can two functions be composed? They are usually given a rule that for two functions f and g, g o f is defined if

(range of f) $\subseteq$ (domain of g). But, then, if g o f and f o g are both defined, is

$f \circ g = g \circ f$? For instance, consider $f : \mathbf{R} \to [0, 1] : f(x) = \dfrac{|x|}{x^2 + 1}$ and

$g : [0, 1] \to \mathbf{R} : g(x) = x$. Then $g \circ f (x) = \dfrac{|x|}{x^2 + 1}$ and $f \circ g (x) = \dfrac{|x|}{x^2 + 1}$.

However, domain (g o f) = $\mathbf{R}$, and domain (f o g) = [0, 1].

So, though it appears that g o f = f o g $\forall$ x, this is not true. This is only true for every x in the intersection of their domains, i.e., [0, 1]. Since the domains of g o f and f o g are not equal, f o g $\neq$ g o f.

Here's a nice exercise for you now.

E11) Ask your learners to do the following exercise.
Give the graphs of two functions f and g chosen by you in a manner that f o g is defined. Also give the graph of f o g. What, if any, pattern do you find that relates the graphs of f, g and f o g?

Analyse their solutions. What insight has this exercise given you about their understanding of the concepts concerned?

There is one type of function that students rarely think of as a function, namely, a sequence. As you know, a real sequence $\{a_n\}_n$ is actually a function $\varphi : \mathbf{N} \to \mathbf{R} : \varphi(n) = a_n$. Students are familiar with arithmetic and geometric sequences, so they can be helped to see these objects as functions. You could do this through exposing them to real-world problems like the following :

Q) *Suppose you have Rs.500/- saved in your drawer at home. Now suppose, you need to help a friend of yours by giving her Rs.50/- each week.*

   i) *How much money will you have at the end of the 3rd week? The nth week?*
   ii) *What type of sequence is being generated?*
   iii) *What function is defined by this sequence, and what are its domain and range?*
   iv) *Graph the function. What shape is the graph?*

Similar questions can be asked for situations that generate geometric sequences, and hence exponential functions.

Why don't you do the following exercise now?

E12) Ask your students to find a recursive formula for representing the following situation. Also ask them Questions (ii), (iii), (iv) above for this situation.

Suppose you hold a ball 2 metres above the ground, and let it go. Each time it bounces, it returns to a height that is 80% of the height from which it started.

Note down the types of errors the students made. Also, talk to them to find out what their reasoning was behind making such errors.

There is a very important aspect of any teaching/learning process that many of us tend to ignore, namely, continuously assessing the level and kind of understanding that the learners have developed. In several studies carried out with children in Class 10 and above, it was found that students had very narrow or erroneous views of what a function is. For instance, many of them believe that

- a function should only be given by a single rule. So, for example, a piecewise function like $f(x) = \begin{cases} 2x, & x \geq 0 \\ -3x, & x < 0 \end{cases}$ is often considered as two functions.

- the graph of a function must be continuous. So, for example, the graph in Fig. 10 doesn't represent a function for them.

- a function is always one-one. So, for example, $f(x) = 0 \; \forall \; x \in \mathbf{R}$ does not define a function because for $x_1 \neq x_2$, $f(x_1) = f(x_2)$. This is due to a confusion arising from the defining property of a single-valued function.

- any algebraic formula will define a function. So, for example, these students believe that $f(x) = \pm\sqrt{x^2 - 1}$ defines a function f, and many believe that $f(x) = 3$ doesn't define a function 'because there is no x on the right-hand side'.
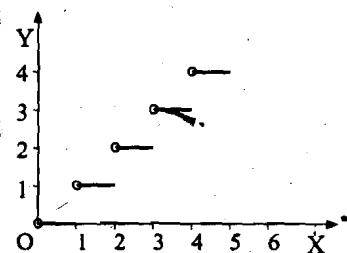
Fig. 10

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This variety of conceptual problems about 'function' has been discovered by giving the students various assessment items to do. This is followed by **interviewing some of the students to find out their reasoning** behind their right or wrong responses, rather than just marking them right or wrong. After all, **errors open windows for us into our students' minds!** They offer us an opportunity to see how the students reason and what their understanding is. We suggest that you do the same with your students, which is what the following exercise is about.

E13) Draw up about 5 or 6 assessment items to give to your students for assessing their understanding of functions. Analyse the results. What are your conclusions about their conceptual clarity of this concept? How different are they from the 4 points we have just given above from the other study?

Now, interview 5 of these students regarding their responses. What insight did this give you about their reasoning about functions, and your teaching method?

In this section, we have spent a fair amount of time on the geometric representations of functions. Let us now consider an interesting aspect of this that is often ignored in our classrooms.

4.4 GRAPHS OF RELATED FUNCTIONS

Let me start with a small anecdote, which may seem unrelated to you. I asked a Class 3 child who had studied multiplication what 18×5 was. She used the algorithm and gave me the correct answer. So I asked her what 18×6 was, and she started doing the algorithm again. She had clearly not been taught to look for relationships between 18×5 and 18×6.

Why I've related this incident is because this is the same problem that our older students face. Do we encourage them to see relationships between the graphs of, say, f and –f ? Or, the graphs of f and $|f|$? Or, the graphs of f and f^{-1}, where f is a bijective function? Since we don't, even though a child may have drawn the graph of x^2 from scratch, she'd start all over again if asked to draw the graph of $x^2 - 10$, rather than just translating the earlier curve. Let us look at what we can do to improve the situation through a few problems of the kind we need to get our students to solve collectively and individually.

Problem 1 : Draw the graphs of the function f, given by $f(x) = x$, and $|f|$ in one diagram. Using the graph of $|f|$, obtain the graphs of g and h given by $g(x) = |x - 1|$ and $h(x) = |x| - 2$.

Solution: Since f and $|f|$ coincide for $x \geq 0$, the graph of $|f|$ is the same as that of f in the first quadrant. Again, since $|f|(x) = -f(x)$ for $x < 0$, the graph of $|f|$ is the reflected image of the graph of f in the x-axis in the second and third quadrants. So, their graphs are as in Fig. 11.
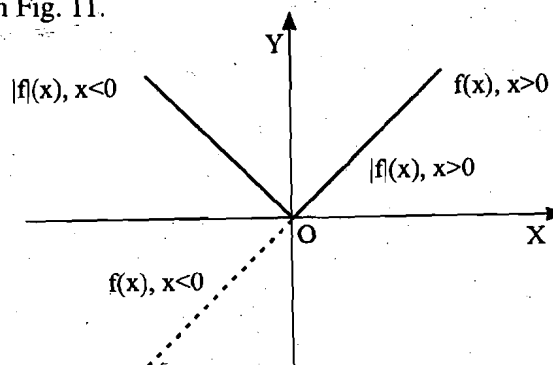

Fig. 11 : The graphs of f and | f |, where f(x) = x $\forall$ x ∈ R.

In Fig.12, we see that the graph of g is obtained from the graph of f just by translating it through 1 unit to the right. Again, the graph of h is obtained from that of | f | by translating it downwards through 2 units.
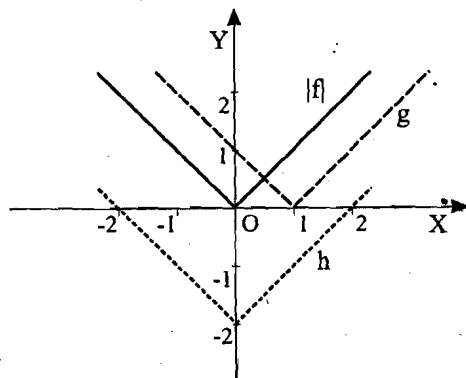

Fig. 12 : The graphs of $|f|$, $g = |f - 1|$ **and** $h = |f| - 2$, **where** $f(x) = x$ $\forall$ $x \in$ **R.**

Problem 2 : Construct the graph of f given by $f(x) = |x + 1| + |x - 1|$, knowing the graphs of f_1 and f_2 given by $f_1(x) = |x + 1|$ and $f_2(x) = |x - 1|$ $\forall$ $x \in$ **R.**

Solution : The idea of this example is to relate the graphs of two functions with the graph of their sum function. So, you can ask your students to construct the graphs of f_1 and f_2 first (as in Fig.13). From these graphs let them find out how to find the graphs of f. Through peer group discussions, or on their own, they may realise that any point on the graph of f is obtained by adding the ordinates of the two graphs at the same abscissa point.
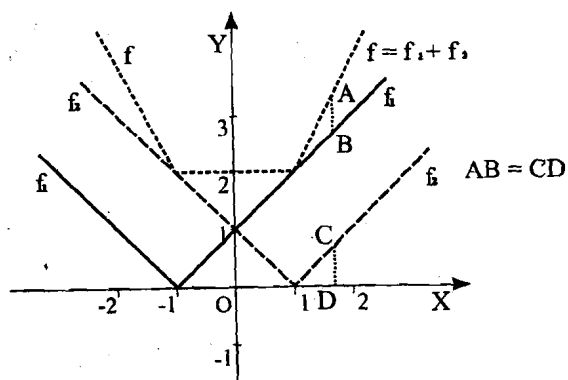

Fig.13

So, to get f from f_1 and f_2, the point A, for instance, is found by adding the ordinate values corresponding to x = OD, which are BD and CD. In other words, they would find the corresponding point on f by adding the vertical length equal to CD at B, which is AB. In this way they would get all the points. (Did they note that at x = 1 and x = −1, f_2 (respectively f_1) has ordinate 0? Hence, they don't contribute to f.)

Problem 3 : Construct the graph of f defined by $f(x) = \dfrac{x}{x^2 + 1}$. Hence construct the graph of g defined by $g(x) = f(|x|) = \dfrac{|x|}{x^2 + 1}$.

Solution : You can ask your learners to apply their knowledge for finding the graph of f (as in Fig.14).
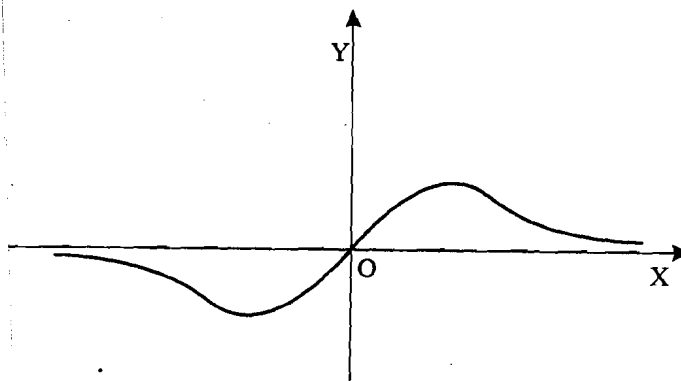
Fig. 14

Next, they should look for cues from this graph for obtaining the graph of g. For this, they need to find relationships between the two functions. For instance, do they see that for x > 0, g(x) = f(x)? If they do, they would know that the portion of the graph of f in the 1st and 4th quadrants will also represent g.

Next, do they note that g is an even function? Using this, they can see that the left half of g is obtained by reflecting the right half in the y-axis. Hence g is as given in Fig.15.
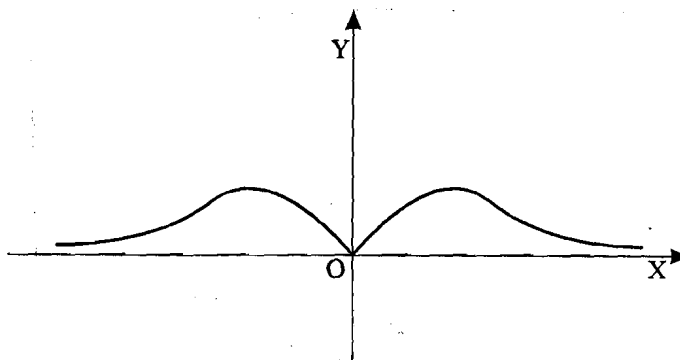

Fig. 15

Why don't you try the following exercises with your learners now?

E14)   Graph the functions f, g, h and θ, defined by f(x) = [x], g(x) = f(|x|), h(x) = f(x + 3), θ(x) = −f(x). Also give the domain and range of each of these functions.

While doing the problem what kinds of difficulties, if any, did your learners face? What methods did you use for resolving their difficulties?

In our interaction with teachers and students, we found that a common problem was how to find whether a function is invertible. And, if it is, then what is its inverse, and how are the graphs of the two functions related? One such interesting discussion is narrated below.

Ms.Azra was interacting with high school teachers. They were discussing trigonometric functions. She drew the graph of the sine function (Fig. 16(a)), and asked them if this function was bijective. One of the participants immediately marked off the portion in the graph restricted to the domain $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, and said that this part was bijective. Then Azra took its reflection in the line y = x, and said that this was now the graph of sin^{-1} restricted to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ (see Fig. 16(b)).
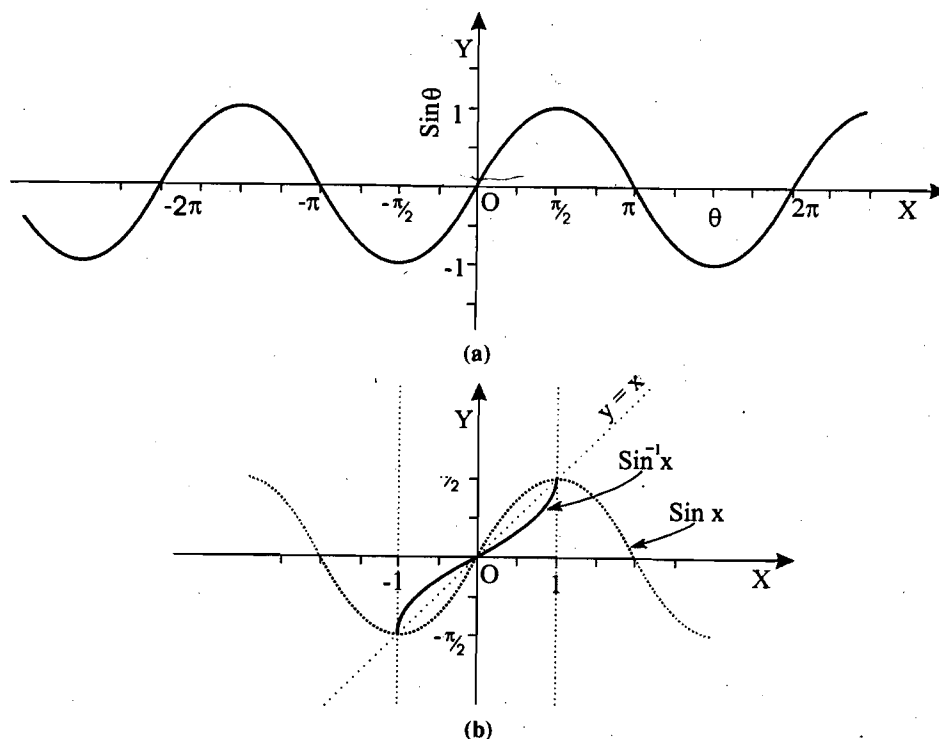
(a)

(b)

Fig. 16

At this point a teacher in the group objected saying, "This can't be the graph of sin⁻¹ because its values are numbers, and what you have along the x-axis are angles."

Do you think this person's objection is valid? Why? How would you explain to him that $[0, \pi[$ is a subset of **R**?

And now let us continue with problems of the kind that our students need to be exposed to regarding connections between f and f^{-1}.

Problem 4 : Given the graph of a bijective function f (as in Fig.17), construct the graph of f^{-1}
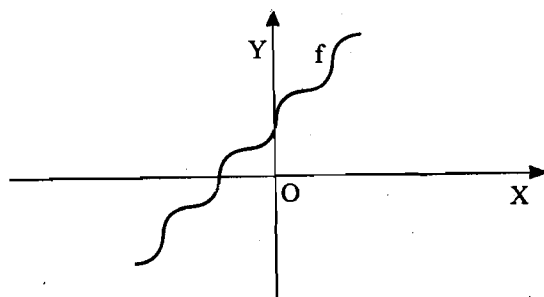

Fig.17

Solution: The inverse g of the function f is the reflection of f in the line y = x. (Why?) So, let us draw both the graphs in Fig.18.
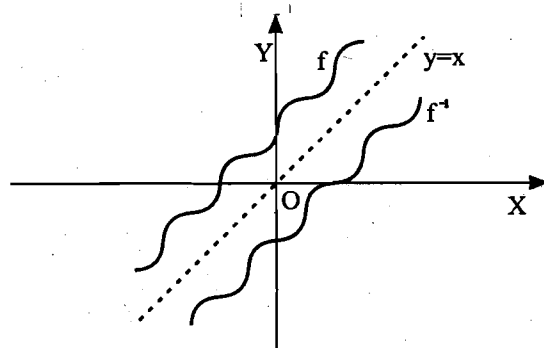

Fig. 18

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Why not try some exercises now yourself, and with your learners?

E15) Find the domain and range of the function f defined by $f(x) = \sqrt{\dfrac{x-3}{x+5}}$.
Check if f is invertible. If it is, define its inverse.

E16) Give an example of a function which is not defined at $x = 4$ and $x = -4$, but bijective everywhere else in **R**. Draw its graph and the graph of its inverse function in the same diagram.

E17) Give examples of three real-life situations which give rise to bijective functions. Also define the inverses of these functions.

With this we end our discussion on certain aspects of sets and functions. Let us take a quick look at what we have covered so far.

4.5 SUMMARY
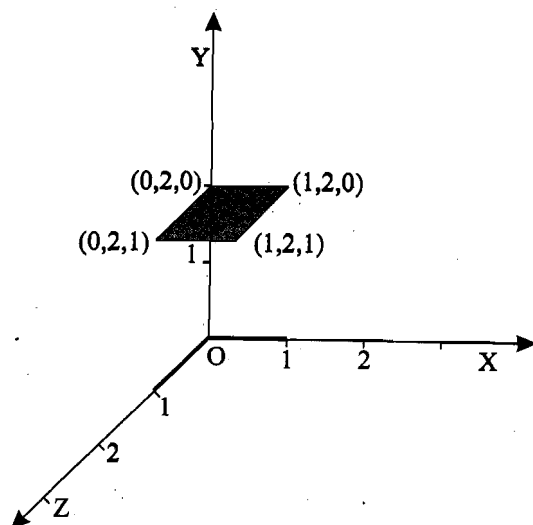
In this unit we have covered the following points.

1) Based on certain misconceptions that students have, we discussed some examples of sets.

2) We looked at some cases where a Venn diagram could not be used as a proof of a statement about sets because it did not cover all the possibilities.

3) We discussed the geometric representation of Cartesian products of sets.

4) We laid stress on using students' real-life experiences for introducing them to any concept, and, in particular, sets and functions. In fact, we gave several examples of activities that teachers have done with their students in this context.

5) We listed several misconceptions students have related to functions, and have asked you to add to this list based on your experiences.

6) We have stressed on developing in the students the ability to graph the sum, difference, modulus, translation, etc., of functions whose graphs are given. Such exposure will improve the students' understanding of functions. Unfortunately, we rarely find such activities being done in our schools.

4.6 COMMENTS ON EXERCISES

E3) We hope you have utilised ideas you thought about while studying Unit 1, when doing this activity.

E4) For instance, all the outcomes obtained when two dice are thrown. Or, all the possible 'dosas' or 'parathas' that can be made with three different stuffings. Think of how these use Cartesian products, and generate other examples.

E5) i) Is it two points, 4 points, or a square?

ii)

Fig.19 : [0,1] × {2} × [0,1]

Note that, though this seems to be three-dimensional, it is actually just the square [0,1] × [0,1] in the plane y = 2.

iii)   This will be two squares.

E6)   Please remember that every error by a student helps you to evaluate the effectiveness of your teaching strategy as well as the quality of mathematical understanding of the student. Did the students' understanding improve when you presented them with your answer to E4?

E7)   An example could be the rule that associates a person with the fruit she eats. Ask your students under what conditions on the people/fruits would this rule be a function, a 1-1 function, an onto function, etc.

Do the same with other examples.

E8)   Over here it would be interesting to find out why your students have chosen a particular example. Did they think of it algebraically first? Or, did they use the vertical line test? Did they define the function, its domain and range clearly? etc.

E9)   For instance, one can define f as follows:

$$f(x) = \begin{cases} 1, & x < -5 \\ 2, & -5 \le x < 0 \\ 3, & x > 5, x \ne 6 \\ 4, & x = 6. \end{cases}$$

More and more complicated examples can be created. But watch out! Their graphical representations could require the use of computer software.

E10)   You could ask different groups of students to graph one function, but use different unit lengths. Then they could compare their outcomes on the board.

Note that the same function can be represented algebraically in different ways also. Give students one or 2 examples of this, and ask them to create other examples of their own. Let them explain to the whole class their choice of example.

E11)   What kind of relationships did they find? In what way has your analysis helped you in improving your teaching of functions?

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E12) The recursive equation: If a_n denotes the height reached after n bounces, then $a_n = (0.8) a_{n-1}$, $a_0 = 2$.

Do your students realise that this is a geometric sequence, and the corresponding function is exponential?

E13) In such an assessment sheet you could include questions like the following:

i) You could give them the graphs of two functions, and ask them to graph their sum or difference.

ii) You could give algebraic representations of several curves like $x^2 + y^2 = 1$, $y = e^{5x}$, etc., and ask them which of these represent y as a function of x, and why.

iii) You could do the same as (ii) above, but with several geometrical representations, of which some would be functions, and some wouldn't.

iv) You could give some real-life situations and ask them if a function shows up anywhere in it, and in what way.

v) You could ask them the following:

An insect is crawling around on a piece of graph paper, as shown alongside (Fig.20). If we wish to determine its location on the paper with respect to time, would this location be a function of time? Why, or why not? Can time be described as a function of its location? Explain.

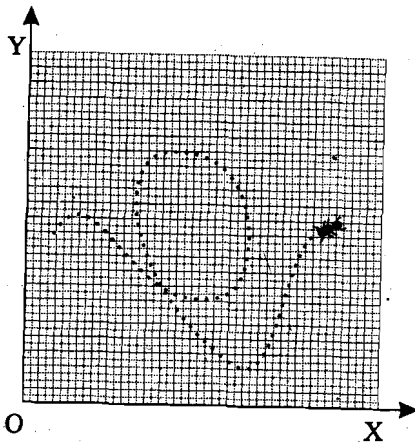You could think of many other interesting tasks to give them.

Fig. 20

E15) Did your students try to graph the function to see if its inverse exists? Or, did they check if it was 1-to-1 algebraically?

E16) As you know, there are infinitely many examples. What kind of variety did you find in your students' answers?

E17) For instance, consider the rule associating a parent with her eldest child. When would such a rule be an invertible function? What would its inverse be?