
UNIT 8 PROBABILITY CONCEPTS

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8.1 INTRODUCTION

One day a friend of mine visited an institute. During her interaction with the people there, she put a few informal questions to them. One question was 'What is the probability of getting a 6 if you throw a dice?' She was taken quite by surprise by the variety of answers she got. Some people said $1/2$, some said $1/7$, some said $1/10$, and only a few said $1/6$. All these people had studied maths till Class 12.

Why are we faced with such a shocking situation? In this unit we look at some of the reasons for this and suggest some methods that may improve the situation.

In Sec. 8.2, our discussion focusses on what 'random' means. We also discuss some problems related to events and outcomes.

In Sec. 8.3 we suggest ways in which you can get rid of the puzzled looks on the face of your learners when asked to find the probability of an event. To do so, we have identified certain common misconceptions in particular, and given some strategies for clearing them.

"Finding the probability is bad enough, but when they ask us to find $P(A|B)$, that takes the cake!" said a student to me. This feeling is common, and we try to understand why in Sec. 8.4. A very important part of this discussion is suggesting ways of helping our learners differentiate between mutually exclusive and independent events.

The scope of this unit is what we have just presented. However, the discussion on probability continues in the next unit, where the focus is on distributions.

Objectives

After reading this unit, you should be able to think of various teaching strategies that interest your learners and develop their ability to

- explain what a **random** experiment is and identify the sample space and events associated with a random experiment;
- identify situations from their environment that require them to find the probability of an event, and to calculate it;
- apply addition and multiplication theorems in the context of dependent events;
- distinguish between mutually exclusive and independent events.

8.2 RANDOM EXPERIMENTS

Many a time, when I ask Class 12 students to explain what a 'random' experiment is, the reply I get is "throwing a coin", or "throwing a dice". "So, if I throw a loaded dice that shows a 6 every time it is thrown, is that a random experiment?" I ask. Usually, after some hesitation, the reply is "No, that is unfair!" So, it appears that 'fair' and 'random' are equated in their minds. On probing further, they usually get flustered. Shouldn't we do something to remedy this situation? The following experience narrated by a teacher may indicate what can be done.

Zahida gives maths tuitions to senior secondary children. To help them understand what 'random' is, she gives them several examples of situations from their daily lives where they need to make non-random and random selections. For example, she asks them that if they are seated in a classroom roll number wise, is the seating random? Through a short discussion she helps them see that since the seating order has been laid down, the outcome is pre-determined. Therefore, the seating is not random. Next, she asks, "What about if you put your hand in a bag of toffees of the same type and draw out one, is that a random choice?" With some prodding they can usually explain that it is random because they can't see what they are drawing out, and all the toffees feel the same. A few more examples of this kind allow them to understand and articulate what 'random' is.

Zahida goes on to relate this with the formal understanding that when a 'random experiment' is performed, the outcome is not pre-determined, i.e., we cannot expect a particular outcome to appear in preference to the others. This is why for the experiment of tossing a coin to be random, the coin must be an unbiased one. Similarly, for the experiment of throwing a die to be random, the die must be a fair one.

In the context of a random experiment, another difficulty that students face is differentiating between events and outcomes, and various ways of describing a single event. In order to help them sort out such confusions, the suggestions in the example below may be of use.

Example 1 : How Ms. Saba introduces her Class 11 students to concepts like sample space, events and outcomes is to divide them into groups. She asks each group to perform random experiments like throwing a 6-faced fair dice, tossing coins, picking out balls blindly from a bag containing several balls which are identical except for colour, and so on. In each case she asks the group to note down what all the possible results are. So, for example, the 'dice' group would have 1,2,..., 6; the 'coin' group would have 'head, tail', etc. She informs them that each result possible is called 'an outcome', and the set of all outcomes is called the 'sample space' of that particular experiment. (Her students are already familiar with sets and subsets.)

Now she asks them what an event means to them. Usually some of them come out with an informal definition like "it is something happening, like a function or show". Here she picks out 'happening', and asks for instance, if in the experiment of throwing a dice, whether getting a 1 is an event, or getting a 6 is an event, or getting an odd number is an event. Through various questions like these, she tries to explain to them that an event is the occurrence of one or more outcomes.

She goes on to connect the outcomes with events through many questions and examples like;

What are the outcomes **in favour** of the occurrence of the event of getting an odd number? With some hints, they do come out with 1,3,5. More questions follow: What are the outcomes in favour of the occurrence of the event of getting a number ≤ 2 ? What is the outcome in favour of the occurrence of the event of getting a 1?

Now is when she tries to connect 'event' with the subset of the sample space consisting of those outcomes which favour the occurrence of the event. For example, the event of getting the number 1 corresponds to $\{1\}$, the event of getting a prime number corresponds to $\{2,3,5\}$, etc. She asks the groups to write down the sample space corresponding to the experiment they are doing, and all its subsets. Sometimes she finds the children forget to include ϕ or the whole set as a subset. Then she asks them to write in words the events corresponding to each subset in the context of the experiment they are performing. For instance, she asks them what $\{H\}$ could be. They say, 'Getting a head.'

Saba finds children have problems about what ϕ could mean, or which event corresponds to the whole sample space. Regarding ϕ , she needs to give them one or two examples of impossible events, and then they come out with several of their own. In the process they also get familiar with the term 'impossible event'.

Regarding giving events described by the whole sample space, she finds many students reply, "Throw the dice till you get all the outcomes." "Is this an event?" she asks. They realise it isn't. "So, give an event which has all the outcomes." With help they come out with one like the event of getting a natural number less than 7. And then, of course, once one example comes out they give many many others. Here is when she familiarises them with the term 'sure event'.

One source of confusion that many students have is that an outcome can't be common to different events. This problem can usually be sorted out by giving them questions like : Which outcomes favour the event that the number appearing when you throw a dice is a prime number? Which outcomes favour the event that the number appearing is an odd number? Is 3 common to both? If 5 is an outcome, which events have occurred? Which have not? And so on.

More generally, Saba helps her students to understand that if E is an event associated with an experiment that is performed, and the outcome is ω , then E has occurred if $\omega \in E$; otherwise, E has not occurred. She does this, again, through exposing them to various questions like — consider the experiment of rolling two fair dice and the event E that the sum of the two scores is 7 or more. What is the sample space and what are the outcomes in E ? Suppose that the experiment is performed and the outcome noted is $(6,4)$. Has E occurred? Has E occurred if the outcome is $(5,4)$?, etc.

You must give a variety of examples and non-examples to your learners to help them develop their understanding of a concept.

Another source of confusion for her learners is about the same event being expressed in various ways. She deals with this similarly, asking them to write, for example, the following events associated with the experiment of throwing a dice:

- i) The event of getting the number 5,
- ii) The event of getting a prime number >3 ,
- iii) The event of getting a multiple of 5.

They realise that each of these events is the same event $\{5\}$. Similarly, they find out that the event of getting a 1 or a 2 and the event of getting a natural number <3 are two different ways of expressing the same event, the subset $\{1,2\}$ of the sample space.

She also asks her learners to do the reverse exercise of describing an event, say $\{2,3\}$, in various alternative ways.

She asks them to do such exercises in the context of a variety of experiments. This is what allows them to construct their understanding.

Ms. Saba has made several points about the teaching-learning of concepts related to experiments. However, one point she didn't touch on, but what we have come across while talking to many teachers, is included in the first exercise below.

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- E1) What is the difference between the experiment of tossing an unbiased coin successively n times and the experiment of tossing n indistinguishable unbiased coins simultaneously? How would you explain your understanding to your learners?
- E2) List at least 3 points made in Example 1 about how children learn?
- E3) Ask your students to specify the sample spaces in each of the experiments described below:
- i) An item produced by a machine is tested to determine whether or not it is defective.
 - ii) A box contains ten cards numbered 1 to 10. A card is drawn, its number noted, and it is replaced. Another card is drawn and its number is noted.
- E4) Suppose a dice is thrown twice. Ask your students to describe each of the following events as a subset of the sample space.
- i) The maximum score is 6.
 - ii) Each throw results in an even score.
 - iii) The scores on the two throws differ by at least 2.
- E5) Sunil is an enterprising Class 8 student who wants to use his free hours more fruitfully. A newspaper agent agrees to employ him for one hour in the morning from 5.30 a.m. to 6.30 a.m. for distributing newspapers in a residential colony where there are 85 regular subscribers. In addition, Sunil finds that there are about 10 irregular customers who may buy the paper from him on a day-to-day basis. On every additional newspaper Sunil sells, he makes an extra income of 30 paise. But on every unsold newspaper that he takes back to the agent he loses 10 paise. Sunil has to decide how many newspapers he should collect from the agent each morning so that he maximises his profits.
- Specify the relevant sample space in the situation above if the random experiment is to observe the 10 irregular customers on any day and note down whether each one "buys" or "does not buy" the newspaper.
- E6) What problems, if any, were faced by your students while doing E3, E4 and E5 above?
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So far, we have looked at student's problems related to the learning of 'random', 'events' and 'outcomes'. Let us, similarly, study the problems related to the calculation of probability.

8.3 WHAT IS THE PROBABILITY?

In your long years of interaction with students, you must have gathered many problems/confusions/misconceptions related to probability that students have. We shall take some of them up one by one in this section.

As you know, probability is a way of measuring the 'chance' of occurrence of an event. Is it more likely to rain today? Is it less likely that Vivek will fall sick on the examination day? Is it more likely that Sita will solve the problem? What is the chance of Ravi getting through the examination? What are the chances of Australia winning the World Cup? While answering such questions, we use probability

implicitly or explicitly. However, when I asked people if they did so, most of them denied it. They told us that the way they were taught probability, just in the context of coins, dice and cards, did not at all translate into developing an ability to think of probability in such real-life contexts.

To improve the situation, we need to bring the students' own experiences into the classroom. All the time, knowingly or unknowingly, they do make use of probability informally to infer the degree of likelihood of the occurrence of an event like those mentioned above. The learners need to be encouraged to **think about and talk about** how they are measuring the likelihood implicitly and how this is connected to the formal calculation of the probability of an event. Only then can their understanding of probability improve.

Even when a student has calculated the probability clearly, she may not have understood what it means. In this context, a common misconception is that if, for instance, the probability of getting a head when a coin is tossed is $1/2$, then in any given number of tosses the outcome of half of them will be heads. So, if 2 tosses are done, 1 will be a head; out of 10 tosses, 5 will be heads, and so on. This misconception stays because we rarely get them to actually do some tosses and see how many heads they get. You need to allow them concrete experiences, from which they can construct their own understanding of why the probability is $1/2$.

It is very interesting to see children doing such **activities**. In one such class, the children were divided into groups. I found the children in one group writing down the probability in the following way.

No. of Tosses	Outcomes	Probability of Head
2	H, H	$\frac{2}{2} = 1$
5	+ H, T, T	$\frac{3}{5}$
10	+ T, H, H, T, H	.
.	.	.
.	.	.
50	+H, H, T, T, T	$\frac{28}{50}$
.	.	.
.	.	.
.	.	.

Other children were doing something similar in their groups. The teacher was trying to find out from them what they thought the probability of getting a head was. At each stage they were getting different figures. So, the general impression seemed to be that it varies. However, the teacher asked them if with more and more tosses, the probability was tending to a certain number? And, what was that number? Here the groups were in unanimity. They were getting nearer and nearer $1/2$. "This is what the probability of getting of getting a head is," said the teacher.

The teacher went on to link this with the axiomatic way of calculating probability. So, without doing all the tosses they could find the probability was $1/2$ in this way. However, she reminded them that, as they had seen, this did not mean that they would definitely get 5 heads in 10 tosses; there could be 8 heads, or only 2 heads, and only in very rare cases, would they actually find 5 heads.

Now, you may like to try the following exercise with your learners.

- E7) Do a similar activity with your students related to drawing balls of a certain colour out of a bag of different coloured balls of the same shape and size. What is the probability that they infer? What were the reactions of the various students to this activity?

Now let's focus on the application of permutations and combinations, for finding the probability of an event. As you know, this is a source of confusion, which we have tried to deal with in the previous unit. Now, consider the following examples linked with many of those in Unit 7. These are the kinds of links you need to help your learners develop, maybe through problems like these.

Problem 1 (Linked with Problem 5, Unit 7): Four cards are drawn at random from a pack of 52 playing cards. Find the probability of getting

- i) all the four cards of the same suit;
- ii) all the four cards belonging to four different suits;
- iii) all the four picture cards;
- iv) two red cards and two black cards;
- v) all cards of the same colour.

Ask your learners: What will be number of possible outcomes in this experiment? What will be the number of those outcomes, which favour the occurrence of the events listed above? Do we need to find the number of permutations or of combinations? Once these numbers are obtained, they should be able to find the probability.

Problem 2 (Linked with Problem 6, Unit 7): Find the probability that in a random arrangement of the letters of the word INDEPENDENCE,

- i) the word starts with P;
- ii) all the vowels occur together;
- iii) all the vowels are not together;
- iv) the word begins with I and ends in P.

Here too, you should ask your learners questions similar to the ones suggested after Problem 1.

Problem 3: Two balls are drawn simultaneously from a bowl containing 6 indistinguishable balls except for colour. 4 of these balls are white and 2 are red.

What is the probability that

- i) both balls drawn are white;
- ii) both balls drawn are of the same colour;
- iii) at least one of the balls is white?

While solving Problem 3, some problems like the one a student, Babu Lal, faced can crop up. According to Babu Lal, the sample space associated with the given experiment was {ww, rr, wr}. When asked for reasons, he said that these were the only outcomes: ww (both white), rr (both red) and wr (one white and one red). Hence,

he promptly answered $\frac{1}{3}, \frac{2}{3}$ and $\frac{2}{3}$ for (i), (ii) and (iii), respectively. In fact, this

confusion is quite common. How would you help the students understand why this is an error?

One way is to get the students to examine this situation. You could ask them to represent the 4 white balls as w_1, w_2, w_3, w_4 and the 2 red balls as r_1, r_2 . Then, in the simultaneous draw of two balls, what will the possible outcomes be? Do they realise that the sample space is $\{w_1 w_2, w_1 w_3, w_1 w_4, w_1 r_1, w_1 r_2, w_2 w_3, \dots, r_1 r_2\}$? How

many outcomes are there? How many of these outcomes favour Event (i)? So, what is the required probability? If they find that it is $\frac{C(4,2)}{C(6,2)}$, your strategy has worked.

Otherwise, it is better to do a similar exercise with several other examples from their own environment, like those related to rainfall, winning games, traffic patterns, etc.

Another common difficulty that students have also shows up while solving Problem 3, that is, 'Should I add or multiply?'. In how many of the possible outcomes are both the balls of the same colour? Will this be $C(4,2) \times C(2,2)$ or $C(4,2) + C(2,2)$? Here the methods discussed in Unit 7 would help.

Now, you may like to try the following exercises yourself and with your learners.

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- E8) Suppose that in a library some of the pages of two of the six copies of a book are sprinkled with ink by the students when they were in use, otherwise from the outside they look exactly alike. If the library assistant selects 2 out of the six copies at random, what is the probability that she will select
- the two damaged copies?
 - at least one of two damaged copies?
- E9) A student takes a multiple choice test composed of 100 questions, each with 4 possible answers. For each question, she blindly picks out one of 4 identical tokens numbered from 1 to 4 from a bag to determine the answer to be marked. What is the probability that she gets the correct answers to 20 questions?
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Let us now look at another source of confusion for children related to finding the probability of the occurrence several events.

8.4 CONDITIONAL PROBABILITY

Let us start with considering the situation of determining the probability of it raining on a particular day. Let us also say that this event, say, A, is influenced by the event B of it raining the day before. We need to take care of this influence while calculating $P(A)$. In fact, this kind of **dependence** of events is very common. The dependence need not be total. The extent of this is reflected in the calculation of conditional probability.

To motivate your learners for studying conditional probability, you could present them with several real-life situations in which the conditional probabilities are known, and we need to calculate the probability of one of the events involved. One such problem is given below:

Problem 4: If it rains today (Event A), the probability that it will rain tomorrow (Event B) is 0.7. If it doesn't rain today, the probability of B occurring is 0.3. Find $P(B)$, if $P(A) = 0.6$.

Solution: We know that $P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$, where \bar{A} denotes 'Not A'. So,

$$P(B) = (0.7)(0.6) + (0.3)(1-0.6) = 0.54$$

One of the reasons students have difficulty in dealing with conditional probability is that they do not understand the problem they are expected to solve. In several cases,

$P(A|B)$ is briefly read as 'the probability of A given B'.

they think conditional probability is required, but are unsure of which one to calculate. And then, sometimes, as in the problems below, the probability can be found directly or by applying the multiplication theorem.

Problem 5 : Suppose an office has 100 computers, some of which are 4th generation (A), while others are 5th generation (B). Also, some are new (N) while others are used (U). The table below gives the number of computers in each category.

Table 1

	A	B	Total
N	20	50	70
U	20	10	30
Total	40	60	100

A person picks a computer at random, and discovers it is new. What is the probability that it is a 4th generation one?

Solution : We want to compute $P(A|N)$. This is $\frac{P(A \cap N)}{P(N)} = \frac{20/100}{70/100} = \frac{2}{7}$.

This could also have been calculated by simply treating N as the reduced sample space (a term you may need to explain to your learners). Then $P(A|N)$ would be $\frac{20}{70} = \frac{2}{7}$.

Problem 6 (Done directly in Problem 3 earlier) : Suppose two balls are drawn without replacement from a bowl containing 6 balls, of which 4 balls are white and 2 are red. What is the probability that

- both balls are white,
- both balls are of the same colour, and
- at least one of the balls is white?

Solution : As in Problem 3, let's number the balls from 1 to 6, where the balls numbered 1 to 4 are white, and the balls numbered 5 and 6 are red. To find the number of possible outcomes, we observe that for the first draw there are 6 choices. Then, since the ball chosen in the first draw is not being put back into the bowl, for the second draw we have 5 choices. So, the sample space, $\{(i,j) | i=1,2,\dots,6; j=1,2,\dots,6; i \neq j\}$ has 6×5 elements.

- Now, let's assume that A is the event that the first ball drawn is white, and B is the event that the second ball drawn is white. Then we need to find

$$P(A \cap B) = P(A) \times P(B|A) = \frac{4}{6} \times \frac{3}{5} = \frac{2}{5}.$$

- Let C and D be the events that the first ball drawn is red and the second ball drawn is red, respectively.

$$\text{Then } P(C \cap D) = P(C) \times P(D|C) = \frac{2}{6} \times \frac{1}{5} = \frac{1}{15}.$$

Now we need to find the probability that either both are white or both are red.

$$\text{This is the sum } P(A \cap B) + P(C \cap D) = \frac{2}{5} + \frac{1}{15} = \frac{7}{15}.$$

- Here, wouldn't the required probability be $1 - P(C \cap D)$? This is so because both balls must not be red. So, the required probability is $\frac{14}{15}$.

Remark : A very important point is made in Problem 6 which needs to be brought to the notice of your students. The point is that the probability of getting both the balls of the same colour in the experiment of drawing 2 balls **one-by-one without**

replacement from an urn is the same as the probability of getting both balls of the same colour in the experiment of drawing 2 balls **simultaneously** from the same urn. In fact, ask your students to do the more general problem:

Problem 7 : Consider an urn containing n balls of which m_1 balls are of Colour 1, m_2 balls are of Colour 2, m_3 balls are of Colour 3, ..., m_p balls are of Colour p , and $m_1 + m_2 + m_3 + \dots + m_p = n$. Now, consider the experiment of drawing m balls from this urn without replacement. What is the probability of getting

- i) all the m balls of Colour 1, where $m < m_1$?
- ii) m balls of the same colour?

Are the students able to arrive at the following answers?

- i) $\frac{P(m_1, m)}{P(n, m)}$, and
- ii) $\frac{P(m_1, m)}{P(n, m)} + \frac{P(m_2, m)}{P(n, m)} + \dots + \frac{P(m_p, m)}{P(n, m)}$.

Do they realise that the answer to (i) is also $\frac{C(m_1, m)}{C(n, m)}$, and to (ii) is the same as

$$\frac{C(m_1, m)}{C(n, m)} + \frac{C(m_2, m)}{C(n, m)} + \dots + \frac{C(m_p, m)}{C(n, m)} ?$$

You may need to be a bit specific here to help them make this connection. For instance, you can ask them to find 2 black, 3 white and 1 red ball from an urn consisting of 4 black, white and red balls each. With more such examples you could move to a more general situation — finding the probability of getting r balls of Colour 1, s balls of Colour 3 and t balls of Colour 5, where $r + s + t = m$? You could give them hints like : What is the number of ways of choosing r balls from m_1 balls of Colour 1? Once they agree that it is $C(m_1, r)$, you could similarly get them to arrive at the number of ways of choosing s balls from m_3 balls of Colour 3 = $C(m_3, s)$, and the number of ways of choosing t balls from m_5 balls of Colour 5 = $C(m_5, t)$. So, what would the total number of ways of choosing $r + s + t = m$ balls be?

$C(m_1, r) \times C(m_3, s) \times C(m_5, t)$, following the multiplication principle. Now they should realise that these will be in the form of m -tuples. Corresponding to each of these tuples, there will be $m!$ arrangements.

Therefore, the number of outcomes favouring the event

$$= C(m_1, r) \times C(m_3, s) \times C(m_5, t) \times m!$$

$$\text{Hence, the required probability} = \frac{C(m_1, r) \times C(m_3, s) \times C(m_5, t) \times m!}{P(n, m)}$$

$$= \frac{C(m_1, r) \times C(m_3, s) \times C(m_5, t)}{C(n, m)}$$

In this way they could understand why we can treat the experiment of drawing balls without replacement as that of drawing m balls simultaneously, provided order does not matter.

Now ask them : What happens if order matters? Give your learners problems like the following to find out.

Problem 8: A bag contains 4 white, 7 black and 5 red balls. Three balls are drawn, one after the other, without replacement. Find the probability that the balls drawn are

- i) in the order of first white, second black and third red;

ii) 1 white, 1 black and 1 red.

Are the two probabilities the same?

Solution : i) Using conditional probability, or other counting arguments, the students can find the required probability = $\frac{4 \times 7 \times 5}{16 \times 15 \times 14}$.

ii) The probability = $\frac{C(4,1) \times C(7,1) \times C(5,1)}{C(16,3)} \neq \frac{4 \times 7 \times 5}{16 \times 15 \times 14}$

Such problems may help them realise that the experiment of drawing m balls one-by-one, without replacement and **in a specific order**, from an urn containing n balls of different colours, **is not the same** as the experiment of drawing m balls simultaneously.

Now, you may like to do the following exercises yourself and give them to your learners. Please encourage the use of diverse solutions.

E10) Suppose an urn contains 6 white, 3 red and 4 black balls. Four balls are drawn from this urn at random one-by-one without replacement. What is the probability of getting 2 white, 1 red and 1 black balls?

E11) A bag contains 10 white and 14 black balls. Two balls are drawn one after another without replacement. Is the probability that the first is white and the second black, the same as the probability of getting 1 white and 1 black ball? Give reasons for your answer.

E12) Prove that $P(A) = P(B)P(A|B) + P(\bar{B})P(A|\bar{B})$ for any two events A and B in a sample space.

Let us now discuss a serious misconception that is commonly found among our learners. This is regarding their inability to **distinguish between exclusive events and independent events**. This confusion can be cleared by exposing the children to a variety of examples in which the events would be independent but not mutually exclusive, and vice-versa.

For example, ask them to consider the experiment of throwing a dice, and look at the following events:

A = the event of getting an odd number = $\{1,3,5\}$;
 B = the event of getting an even number = $\{2,4,6\}$.

Will the occurrence of one rule out the occurrence of the other? Yes, indeed, because, they don't have any element in common. Through a Venn diagram also they should see why A and B are mutually exclusive events. Then they should find out why $P(A \cap B) = 0$.

On the other hand, ask them what $P(A|B)$ and $P(B|A)$ are? Do they realise why $P(A|B) = 0$? (Because if we assume that B has occurred, there is no question of the occurrence of A .) This is also why $P(B|A) = 0$. So, the occurrence of one heavily influences the probability of the occurrence of the other. Therefore, they are certainly not independent.

Now, give them the following problem:

Two balls are drawn at random **with replacement** from a bowl containing 4 white and 2 red balls. What is the probability that both balls are white?

You could ask them to name the events concerned as below.

A = the first ball drawn is white;

B = the 2nd ball drawn is white.

Do they see that A and B are independent?

Do they link this with $P(A \cap B) = P(A) \times P(B|A) = P(A) \times P(B)$?

Now ask them if A and B are mutually exclusive. Asking them to write down A and B in subset form may help. What are the outcomes of A?

$A = \{(i, j) \mid i = 1, 2, 3, 4; j = 1, 2, 3, 4, 5, 6\}$, where the white balls are numbered from 1 to 4 and the red balls are numbered 5, 6.

What are the outcomes belonging to B?

$B = \{(i, j) \mid i = 1, 2, 3, 4, 5, 6; j = 1, 2, 3, 4\}$.

So, is $A \cap B = \emptyset$? What can you conclude about A and B being mutually exclusive?

Remark : In the example above, had we drawn the second ball without replacing the first one, would $P(A|B)$ be meaningful? In fact, $P(A|B) = P(A)$ as the occurrence of B has no effect on the probability of A. Though, $P(B|A)$ would not be $P(B)$.

You may like to try the following exercises now.

E13) Suppose an unbiased coin is tossed twice. Let F be the event that the first toss results in a head and E be the event that the second toss produces a head. Prove that E and F are independent but not mutually exclusive.

E14) Prove that if two events A and B are independent, then A and B cannot be mutually exclusive unless at least one of them is an impossible event.

We shall end this discussion on learning difficulties related to finding the probability here. However, our discussion on probability distributions continues in the next unit. Before we end, let us take a quick look at what we have discussed here.

8.5 SUMMARY

Our learners have several difficulties related to the learning of basic probability. Some of them have been dealt with in this unit. In particular, strategies have been suggested for helping them understand the following points.

1. An experiment is called 'random' if we cannot predict which outcome we will get on performing it.
2. Any subset of the sample space is an event. An event could be expressed in various ways.
3. Examples of finding the probability of events related to the students' own environment.
4. The experiment of tossing a coin n times can be treated as tossing n identical coins simultaneously.

5. The difference between mutually independent events and mutually exclusive events.

8.6 COMMENTS ON EXERCISES

- E1) The two are the same. For explaining this to your learners, in the latter case, they may have to number the n coins as 1st, 2nd, 3rd, ..., n th for the sake of obtaining the possible outcomes of the experiment. Ask them to give an example of an outcome in this experiment. One of them could be (H, T, H, H, ..., T), where the results recorded on the 1st, 2nd, 3rd, 4th, ..., n th coin, respectively, are H, T, H, H, ..., T. How many possible outcomes can they form? Is it 2^n ? If so, why?

Similarly, ask the students: is the experiment of throwing a fair dice successively n times the same as the experiment of throwing n fair indistinguishable dice simultaneously?

- E2) For instance, allow children to discover the concept themselves through exploration. List other points. You may also like to look at Unit 1 while doing so.

- E3) i) $S = \{g, d\}$, where g stands for 'non-defective' and d stands for 'defective'.
 ii) Suppose we code the ten cards by the numerals 1, 2, 3, ..., 10. Then $S = \{(x, y) \mid 1 \leq x \leq 10, 1 \leq y \leq 10 \text{ and } x, y \in \mathbf{N}\}$

- E4) Suppose i is the result on the first throw and j is the result on the second throw. $i = 1, 2, \dots, 6$ and $j = 1, 2, \dots, 6$. Thus, the sample space is as follows:
 $S = \{(i, j) \mid i = 1, 2, 3, \dots, 6; j = 1, 2, \dots, 6\}$ or $S = \{1, \dots, 6\} \times \{1, \dots, 6\}$
 i) $\{(i, j) \in S \mid i + j \leq 6\}$
 ii) Event that each throw results in an even score = $\{(i, j) \mid i = 2, 4, 6 \text{ and } j = 2, 4, 6\}$
 iii) Event that the scores on the two throws differ by at least 2 = $\{(i, j) \mid i = 1, 2, \dots, 6; j = 1, 2, \dots, 6 \text{ and } |i - j| > 1\}$

- E5) Since an irregular customer may or may not buy the newspaper, an outcome will be of the form of a 10-tuple $(a_1, a_2, a_3, \dots, a_{10})$, where each a_i is either 'B' or 'NB', and B represents 'buys', NB represents 'does not buy'.

Hence, the sample space $S = \{a_1, a_2, a_3, \dots, a_{10} \mid a_i = B \text{ or } NB, i = 1, 2, \dots, 10\}$

$|S| = 2^{10}$ because each a_i has 2 possibilities, B and NB.

- E6) What methods did you use for finding out the problems? How did you remedy the situation?
- E7) How many trials of the experiments did each group do? Were they able to arrive at the correct probability? How many of them were interested in this approach? How many were confused with the process?
- E8) i) The probability that she selects the two damaged copies

$$= \frac{C(2,2)}{C(6,2)} = \frac{1}{15}$$

$$\begin{aligned}
 \text{ii) The probability that she selects at least one of the two damaged copies} \\
 &= 1 - \text{the probability that she selects non-damaged copies} \\
 &= 1 - \frac{C(4,2)}{C(6,2)} = 1 - \frac{2}{5} = \frac{3}{5}.
 \end{aligned}$$

Note: You may also treat the experiment above as the experiment of drawing two copies from out of 6 copies one-by-one without replacement, and apply the multiplication theorem to obtain the required probability.

- E9) Obviously, the total number of ways in which she can answer all the 100 questions is $4 \times 4 \times 4 \times \dots$ (100 times) $= 4^{100}$. The 20 questions to be answered correctly can be picked in $C(100,20)$ ways. Corresponding to each of these selections, she can answer the 80 remaining questions in 3 ways (the 3 wrong answers). The 20 questions can be answered correctly in only one way. Hence, the number of ways in which 20 questions are answered correctly $= C(100, 20) \times 3^{20} \times 1^{20}$.

$$\text{Therefore, the required probability} = \frac{C(100,20) \times 3^{20} \times 1^{20}}{4^{100}}$$

$$\text{E10) The required probability} = \frac{C(6,2) \times C(3,1) \times C(4,1)}{C(13,4)}$$

$$\text{E11) } P(\text{first is white and second is black}) = \frac{10}{24} \times \frac{14}{23} = \frac{35}{138}$$

$$\text{On the other hand, } P(1 \text{ white and 1 black}) = \frac{C(10,1) \times C(14,1)}{C(24,2)} = \frac{35}{69}$$

(treating the experiment the same as that of simultaneous draw).

- E12) You could use the Venn diagram approach and multiplication theorems to arrive at this.

$$\text{E13) } S = \{(H,H), (H,T), (T,H), (T,T)\}$$

$$E = \{(H,H), (T,H)\} \text{ and } F = \{(H,T), (H,H)\}$$

$$E \cap F = \{(H,H)\} \neq \phi. \therefore E \text{ and } F \text{ are not mutually exclusive.}$$

$$P(E) = \frac{2}{4} = \frac{1}{2} = P(F)$$

$$P(E \cap F) = \frac{1}{4} = P(E) \times P(F)$$

$\therefore E$ and F are independent.

- E14) If A and B are independent, $P(A \cap B) = P(A)P(B)$.

Now, if they are exclusive, then $P(A \cap B) = 0$

$$\Rightarrow P(A)P(B) = 0 \Rightarrow P(A) = 0 \text{ or } P(B) = 0 \Rightarrow A = \phi \text{ or } B = \phi.$$