UNIT 7 COUNTING TECHNIQUES

Structure **Page Nos.**

7.1 INTRODUCTION

In this unit, we focus on an area you usually teach to Class 11 students. Here we aim to discuss ways of presenting some counting techniques to your learners. As you know, finding the number of permutations and combinations in a situation allows us to answer 'how many?' without actually counting.

The counting techniques that we shall discuss here have a long history, which is very nicely given in the NCERT Class 11 textbook. We find that in India the use of permutations and combinations can be traced back to mathematicians such as Mahavira (ca 850 AD) and Bhaskaracharya (12^{th} century AD), the doctor Sushruta (6^{th} century BC), and the linguist Pingala (3^{rd} century BC), among others.

Our discussion in the unit begins with a brief look at some points of confusion that children face regarding the use of the multiplication principle. Then we consider some ways of exposing your learners to a variety of real-life problems related to permutations and/or combinations.

The reason we have included this unit in a block on probability may not be obvious to you. But, what we do here is closely linked with the next unit, as you will see by the end of Unit 8.

Let us now list the objectives of this unit.

Objectives

After studying this unit, you should be able to help your learners develop the ability to

- explain why the multiplication principle is appropriate in a particular situation;
- identify when a given problem requires the number of permutations to be counted, and when it requires the combinations to be counted;
- describe permutation as a function.

7.2 MULTIPLICATION OR ADDITION?

Let us start by assuming that you want to introduce your students to permutations and combinations. Would you first want them to be familiar with the multiplication principle? How would you assess if they know it already? Probably by using the method suggested in Example 1.

Example 1 : A senior school teacher began his series of lessons on permutations and combinations by trying to find out whether his students were ready for these concepts. He started the class with informally asking them to do the following problems.

Q1) *A man has 3 shirts and 2 ties. How many different shirt-and-tie combinations can he wear using these?*

Q2) *I go out to a dinner at a restaurant which provides the following menu :*
Starter : Soup or salad
Main dish : Rajma-rice, chicken biryani or mutton biryani
Sweet dish : Cake or ice-cream.
If I order a starter, a main dish and a sweet dish, in how many different ways can I do this?

The teacher found that quite a few children in his class could solve both questions. He asked one of them to explain to the other children the way he had adopted. The child was used to explaining his solutions. He promptly came to the board and drew a diagram (see Fig. 1) on it, explaining as follows:

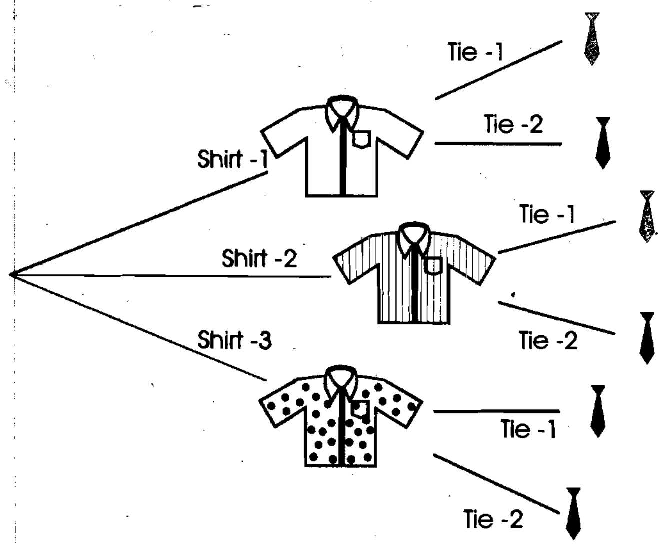

Fig. 1 : Tree diagram corresponding to Q.1

"There are 3 ways of choosing a shirt from out of 3 shirts, say, Shirt 1, Shirt 2, Shirt 3. Then, for each shirt, there are 2 ways of choosing a tie, say, Tie 1, Tie 2. So, total 6 ways."

The teacher, then, asked the class if the diagram covered all the possibilities. The students agreed. So, the teacher ended this discussion with writing

3 shirts and 2 ties have $3 \times 2 = 6$ possibilities

on the board.

Next, the teacher asked another child to explain how she did Q. 2. This student said her answer was 2+3+2 ways. Her reason was that this was the total available to the diner. This reply generated a discussion in the class till another child came to the board, and drew a diagram (see Fig. 2).

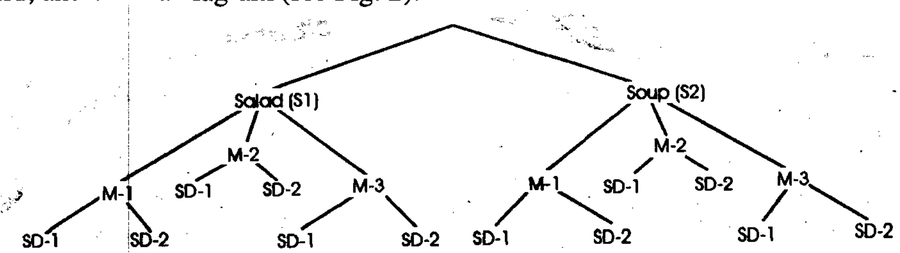

Fig 2 : Tree diagram corresponding to Q.2

Then she used the diagram to explain her own understanding that three separate decisions have to be taken to order the meal — choosing a starter, choosing a main dish and choosing a sweet dish. So, the diner could order S1, M1, SD1, or S1, M2, SD1, and so on. So, there are a total of $2 \times 3 \times 2 = 12$ possibilities.

Some children were still not convinced. The teacher, then, asked them to write down all the options and count them. Then he went back to the tree diagram to convince them why the number of possibilities were multiplied. Finally, these children understood the reason in this situation.

At this stage, the teacher asked them "suppose, in Q.2, the situation now changes. Now I can either eat a starter, or a maindish, or a sweet dish. In how many different ways can this be done?" The students thought for a bit, discussed with each other, and came out more or less unanimously with the answer "same as earlier, 12". He firmly told them to think again, and gave them some hints — suppose there were only starters, how many different dishes could I have?, and so on. Finally, the students came out with the solution — adding all the possibilities, i.e., 2+3+2 = 7.

The teacher followed this discussion with giving the children a few more problems to do some involving multiplication and some addition ones. A few children still remained unsure about whether to add the number of options at each stage, or multiply them, and why. At this stage, the teacher took these children through each problem, explaining case by case that for each of the m possibilities of the first event, there were n possibilities for the second event. Gradually, step-by-step the children were exposed to a general pattern emerging from several particular cases. By the next class they seemed to have developed an understanding of why the counting requires the use of multiplication or of addition.

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Having gone through this example, here is an opportunity for you to reflect on your own classroom interactions.

E1) Can the fundamental principle of counting be introduced to learners in earlier classes, say, in Class 10, or in Class 6? If yes, what strategy would be use for the Class 10 children? Would children of Class 6 understand by this method? If not, how would you introduce them to this principle?

If you believe that children are not ready for the multiplication principle before Class 11, please give your reasons.

Let us now discuss issues related to the learning of 'the Ps and the Cs'!

7.3 PERMUTATIONS OR COMBINATIONS?

A common question students ask themselves when faced with a problem like 'In how many ways can a principal and vice-principal be appointed from amongst a group of 10 teachers?' is :

Should I find P(10,2) or C(10,2)? Here is how Mandeep Kaur, a resource group teacher, helps her learners in this regard.

Example 2 : Mandeep Kaur introduces her students to permutations and combinations through problems related to their daily existence. She prefers to introduce these concepts more-or-less simultaneously so that the students understand the difference in the situations that require them to find P(n,r) and C(n,r). **Once the students**

understand what a permutation is and what a combination is, she gives her learners problems like 'Find the number of 5-letter words, with meaning or without meaning, which can be formed out of the letters of the word LOGARITHMS, where the repetition of letters is not allowed.'

How would your learners try to solve the problem? Would they list out all the words such as LOGAR, OLGAR, LOGAI, ... etc., and count them? How many words would they list out before they realise that this process of counting would be difficult? Mandeep finds that her students tend to do this, and beyond a point, start getting bored.

Here is where Mandeep tells them that there is a short-cut, and asks them to think, in groups, of what it could be. Then she drops a hint here and a hint there — Would you agree that the word LOGAR is different from OLGAR? The order of the letters is important. Are you counting permutations then? If repetition is not allowed, what is the required number of words? Note that the word LOGARITHMS contains 10 letters.

She also asks the students several questions to help them link this with the number of permutations of 10 letters taken 5 at a time — How many possibilities does the first letter have? Then, how many does the second letter have? And so on. Finally, the students do agree that the total number $= 10 \times 9 \times 8 \times 7 \times 6 = 30240$ (applying the multiplication principle), and that this number is the same as ${}^{10}P_5$ or $P(10, 5) = \frac{10!}{(10-5)!}$.

Now she asks them that if the repetition of the letters was allowed, what would the number of words be? Usually, most students realise that each of the 5 letters has 10 possibilities. So, the required number would be $10 \times 10 \times 10 \times 10 \times 10 = 10^5$.

Now, Mandeep takes them further, with more examples like the one we mentioned at the beginning of the section. While discussing that one, she asks if order is important. Is the choice of Ms.Annam and Mr.Batra as the principal and the vice-principal, respectively, different from the choice of Mr.Batra and Ms.Annam as the principal and the vice-principal, respectively? Will the number of ways be P(10,2) or C(10,2)? Here some students tend to use the basic counting principle that the principal can be chosen in 10 ways, and then the vice-principal in 9 ways, so, the total number of possibilities is $10 \times 9 = \frac{10!}{(10-2)!} = P(10,2)$; and therefore, the answer is P(10,2)!

Others realise that this is a situation requiring the counting of permutations, and get P(10,2) straightaway. And then, there are some who reason that two people have to be chosen from 10, so it is C(10,2). In such a situation, it is clear to Mandeep that they need some more exposure to a variety of problems for developing the level of understanding she wants them to have.

For this, Mandeep usually tells them, "Let's assume, now, that there is a group of 3 table tennis players, namely, Shiraz, Mohan and Javed. A team consisting of 2 players is to be formed. In how many ways can we do so? Think about whether the team of Shiraz and Mohan is different from that of Mohan and Shiraz."

The children soon realise that order doesn't matter. So, there are three possible ways in which the team could be constructed :

Shiraz, Mohan

Shiraz, Javed

Mohan, Javed

When she asks some children to explain their solutions, the answers at this stage usually are "Here, in this case we have to select two players out of the given 3 players, and the ordering of the chosen players does not matter. So, each selection is a combination (*a term she had introduced them to in earlier lessons*). So we need to find the number of combinations here, not permutations. So, the number will be ${}^3C_2$ or C(3, 2) = 3."

Next, Mandeep asks, "Now suppose we want to give two prizes to two out of the three of your friends, Radhika, Vivek and Priyanka. In how many ways can we do this?". The children think about this, discuss it, and usually conclude that the order doesn't matter. So, the number of ways would be C(3,2).

Then Mandeep asks, "If we had a first and a second prize to give to them, then in how many ways can we do this?". In this case the children do realise that order would matter. So, they would need to count the number of ways to award the prizes. So, in this case, they would need to count the permutations of 3 people taken 2 at a time. Some children also draw a chart like the following one to help them count.

First Prize	Second Prize
Radhika	Vivek
Vivek	Radhika
Radhika	Priyanka
Priyanka	Radhika
Vivek	Priyanka
Priyanka	Vivek

So, the required number of ways in this case would be P(3,2) = 6, they conclude.

Off and on, Mandeep assesses how far the students can gauge whether a situation requires the number of permutations to be counted or the number of combinations. So she exposes them to a series of mixed problems for discussion in class and at home. Some of the 'more challenging ones' are given below along with the solutions that the children came up with, either on their own, or with hints from her.

Problem 1 : How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5 if the repetition of digits is not allowed?

Solution : Every number between 100 and 1000 is a 3-digit number. But, we need to count those 3-digit numbers which are formed with the digits 0,1,2,3,4,5, keeping in mind that repetition of the digits is not allowed. Here order matters because, for example, 103 and 130 are two distinct numbers. So, we need to count the permutations of 6 digits taken 3 at a time. (*Here Mandeep often needs to remind them that these P(6,3) permutations will include those which have 0 in the 'hundreds' position, and these are not 3-digit numbers. She also asks : How should we find the number of such permutations?*) All such numbers with 0 at the hundreds place are got by rearranging the remaining 5 digits taking 2 at a time. This number is P(5,2).

So, the required number that the problem asks for is

$$P(6,3) - P(5,2) = \frac{6!}{3!} - \frac{5!}{3!} = 4 \times 5 \times 6 \; - \; 4 \times 5 = 100.$$

Some children solved this by noting that the number of possibilities in the hundreds place is 5; then once this is fixed, the number of possibilities in the tens place is again 5; and finally, the number of possibilities in the units place is 4. So, by using the multiplication principle, they obtained the answer, $5 \times 5 \times 4 = 100$.

Problem 2 : A committee of 5 persons is to be constituted from a group of 6 men and 8 women. In how many ways can this be done? How many of these choices would consist of 3 women and 2 men?

Solution : To start with, the students ask themselves : Is a change in the order of 5 persons selected from out of 14 persons going to change the committee? Once they decide the answer is 'No', they know that they need to count the number of combinations. Hence, the required number of ways = $C(14,5) = \frac{14!}{5!\,9!} = 2002$.

Now, for the second part of the problem, 3 women can be selected from 8 women in C(8,3) ways. Similarly, 2 men can be selected from 6 men in C(6,2) ways.

"So, in how many ways can a committee consisting of 3 women and 2 men be formed?", asks Mandeep. Here many students do get confused — C(8,3) + C(6,2) or C(8,3) × C(6,2)? So, Mandeep explains that the committee is formed if corresponding to each choice of C(8,3) ways, a choice of C(6,2) ways is made, which she shows them diagrammatically as given in Fig. 3. This, with several questions from Mandeep, helps the children realise that the multiplication principle is appropriate for finding the required number of ways.)

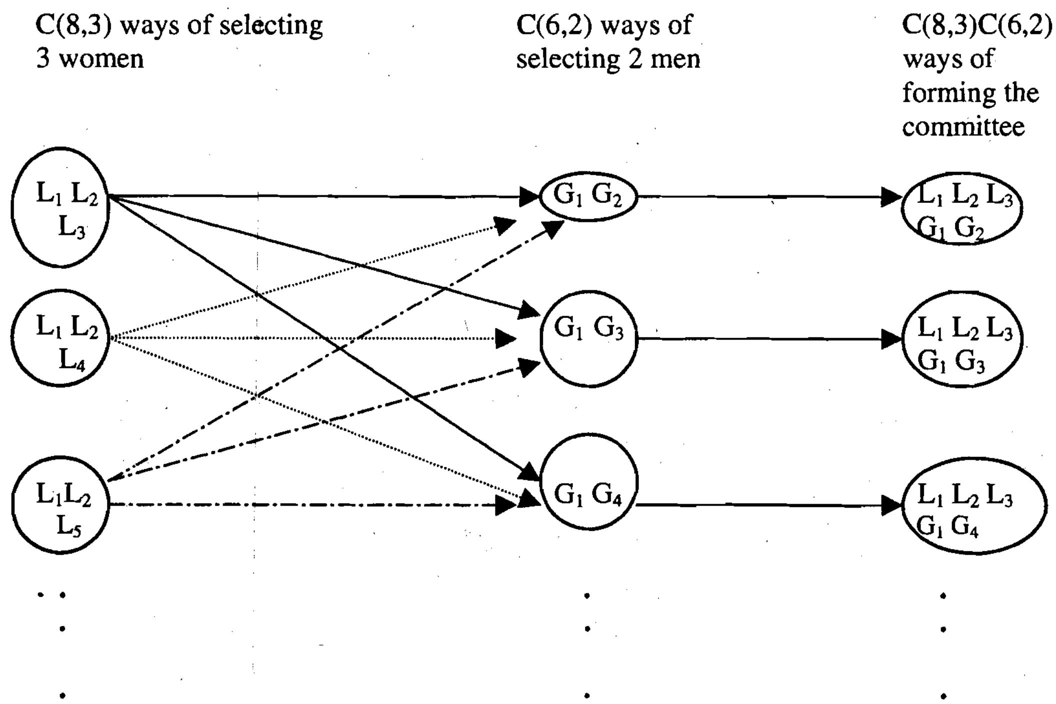

Fig. 3

Therefore, the total number of ways a committee consisting of 3 women and 2 men can be constituted $= C(8,3) \times C(6,2) = \frac{8!}{3!\,5!} \times \frac{6!}{2!\,4!} = \frac{6\times7\times8}{2\times3} \times \frac{5\times6}{2} = 840$.

Problem 3 : There are 8 candidates appearing in a test, 3 of them for mathematics and the remaining 5 for 5 different non-math subjects. How many seating arrangements are possible if they are to sit in a row with no two mathematics students sitting together?

We must encourage students to come out with diverse solutions.

Solution : *(Some students decided to first find out in how many ways the 5 students appearing in 5 different subjects can sit. They found this number and thought that the task was complete. Others realised this wasn't so because corresponding to each of these arrangements, the three mathematics students had to be accommodated.)*

The 5 non-math students can be seated in P(5,5) = 5! ways. Corresponding to each of these ways, the 3 math students can occupy **any three of the 6** cross-marked places shown below :

$$X - X - X - X - X - X.$$

The number of permutations of these 6 spaces filled 3 at a time gives the number of ways in which the three maths people could be seated. This is P(6,3).

So, corresponding to each of the 5! ways in which the other students can sit, the math students can sit in P (6,3) ways.

So, to find the total number of seating arrangements required, we use the multiplication principle, to get $5! \times P(6,3) = 14400$.

Remark : Some students tried another path for solving this problem as follows. They said there are eight seats in a row, say,

(1) (2) (3) (4) (5) (6) (7) (8).

Now (1) can be filled by a math or a non-math student. So, if a math student takes it, (2) will be taken by a non-math student. This can be done in 3×5 ways. Next, (3) can be filled by a math or a non-math student, and so on. However, the number of possibilities were becoming too difficult for the students to keep track of. And they gave up half-way.

Problem 4 : A bag contains 30 tickets, numbered 1 to 30. Five tickets are drawn at random and arranged in ascending order. In how many of the possible arrangements is the third number 20?

Solution : Each arrangement, where the third number is 20 will be of the form:

$a_1, a_2, 20, a_4, a_5$, where $a_1, a_2 \in \{1, 2, 3, 4, \ldots, 19\}$ and $a_4, a_5 \in \{21, 22, \ldots, 30\}$.

Therefore, the required number of arrangements = $C(19,2) \times C(10,2)$

$$= \frac{19!}{2!17!} \times \frac{10!}{2!8!} = 7695.$$

Problem 5 : What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these are

i) all the four cards of the same suit?
ii) all the four cards belonging to four different suits?
iii) all four picture cards?
iv) two red cards and two black cards?
v) all cards of the same colour?

Solution : The number of ways of picking 4 cards from a pack of 52 cards is C(52,4). *(The students got this quite easily.)*

i) There are four suits : spade, heart, diamond and club. There are 13 cards of each suit. Therefore, there are C(13,4) ways of choosing 4 spades, C(13,4) ways of choosing 4 hearts, C(13,4) ways of choosing 4 diamonds, C(13,4) ways of choosing 4 clubs.

(So, what will the total number be? Here is where many students again get confused. Very often all of them write $C(13,4) \times C(13,4) \times C13,4) \times$

C(13,4).Mandeep tries various ways of addressing this confusion. What seems to work well is using a Venn diagram to explain that there are 4 sets of possibilities, and these sets are mutually exclusive. Therefore, all the elements of these sets would give all the possibilities. So, all of these should be counted.)

Hence, the required number of ways = C(13,4) + C(13,4) + C(13,4) + C(13,4)

$$= 4 \times \frac{13!}{4!\,9!} = 2860.$$

ii) There are C(13,1) ways of choosing one card from one suit. Since the 4 cards have to belong to four different suits, the required number of ways is $C(13,1) \times C(13,1) \times C(13,1) \times C(13,1) = 13^4$.

iii) There are 3 picture cards in each suit, and 4 are to be selected out of these 12 cards. This can be done in C(12,4) ways.

Therefore, the required number of ways $= \frac{12!}{4!\,8!} = 495$.

iv) There are 26 red cards and 26 black cards. There are C(26,2) ways of selecting the red cards and C(26,2) ways of selecting the black cards. Therefore, the required number of ways $= C(26,2) \times C(26,2)$

$$= \left(\frac{26!}{2!24!}\right)^2 = (325)^2.$$

v) There are C(26,4) ways of selecting all four red cards, and there are C(26,4) ways of selecting all four black cards. Therefore, the required number of ways

$$= C(26,4) + C(26,4) = 2 \times \frac{26!}{4!22!} = 29900.$$

Ms. Mandeep Kaur had a lot more to share with us about the variety of reactions she gets from her students regarding the connections she helps them make between these counting techniques and finding the probability of an event. But we shall leave that for now.

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Having gone through Example 2, you may like to try the following exercises yourself and with your students. Also, while you are doing the exercises, please note down your own thought processes and the steps you go through. These are the steps that your learners would also go through, may be going back and forth between them more often than you do.

E2) If n couples are at a dance, in how many ways can the men and women be paired for a single dance?

E3) Consider all the numbers between 100 to 999 that have distinct digits. How many of these are odd?

E4) How many 5-digit integers greater than 65000 have both the following properties?

i) the digits of the number are distinct;

ii) the digits 0 and 1 do not occur in the number.

E5) While doing E2 to E4 above, what kind of errors and confusions did the students have? What was their reasoning behind these errors?

So far we have discussed various situations that children need to be exposed to for developing their conceptual and procedural understanding of permutations and combinations. As part of their syllabus, they are also expected to study particular types of permutations. These types are covered in the NCERT course, but rather cursorily. We shall now discuss them in a little more detail.

7.4 SOME MORE ABOUT PERMUTATIONS

In this section we shall focus on two types of permutations that we have not discussed so far.

We shall also discuss an aspect of permutations that you may not have thought of so far, namely, permutation as a function.

7.4.1 Permutations When All The Objects Are Not Distinct

Suppose you ask your students to find the number of ways of rearranging the letters of the word SCHOOL. How did they do this? Did they list out all the possibilities? Or did they answer 6! because there are 6 letters in this word? Did they realise that two of the letters are the same? To help your students understand why the required number is $\frac{6!}{2!}$, you could explain as follows:

Let the required number of ordered arrangements (permutations) be x. If the letters were all different, the total number of permutations would indeed be 6!. To understand why this situation is different, consider one of the x permutations, say, SOCHOL. If the two Os are treated as two different letters O_1 and O_2, then the permutation SOCHOL is actually 2! different permutations, namely, SO_1CHO_2L and SO_2CHO_1L.

So, each of the x permutations gives rise to 2! different permutations since the two letters O_1 and O_2 can be rearranged in 2! ways.

Therefore, the total number of permutations, treating the two Os as different, is $x \times 2!$. On the other hand, we have already noted that this number should be 6!..

So, $x \times 2! = 6!$, which implies, $x = \frac{6!}{2!} = 360$.

To see if they have understood your explanation, you could ask them what the number of arrangements of the letters of the word BHAGALPUR will be. You should also ask a few students to explain to the others why this number is 181440.

Gradually, you could move to the more general case. This is what the following question is about.

E6) How would you explain to your learners that the number of permutations of n objects, p of which are the same, is $\frac{n!}{p!}$?

Now, suppose you go a step further, and ask your students to try the following problem:

What will the number of arrangements of the letters of the word ALLAHABAD be?

Here, there are 9 letters, of which 4 letters are of one kind (As), 2 letters are of another kind (Ls), and the rest are all different. They may very well solve it in the following manner:

If x is the required number of permutations, and if all the 4 As are treated as different, then each of the x permutations will give rise to $x\times4!$ permutations, as the 4 As can be rearranged in 4! ways. Further, if both the Ls are treated as different, then each of the $x\times4!$ permutations will give rise to $x\times4!\times2!$ permutations.

Therefore, the total number of permutations, taking the 4 As and the two Ls as different $= x\times4!\times2!$.
On the other hand, this number is also 9!.

$\therefore 9! = x \times 4! \times 2!$

$$\Rightarrow x = \frac{9!}{4!2!} = \frac{5\times6\times7\times8\times9}{2} = 7560.$$

If they come up with this solution, then their understanding of the situation seems to be fine. Maybe, you can provoke them to consider the more general situation now. What will the number of permutations be of n objects, p of which are of one kind, q of them are of another kind, and $n-p-q$ of them are all different?

Were they able to arrive at the answer, $\frac{n!}{p!q!}$? If so, what were the different proofs they came out with? Did they discuss their solutions with each other? Did you encourage them to generalise the situation further?

Why don't you try an exercise in this direction now?

E7) What is the most general result you can state regarding the number of permutations of objects in which all of them are not distinct? Also prove your Result.

During your interaction with your learners, you would be evaluating their grasp of the concepts continuously. How do you go about it? One way is to

give your learners problems of the following kind to solve. This may help to reinforce their conceptual understanding. (We have also given the solutions that several students have given, with or without their teachers' help.)

Problem 6: Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,

i) do the words start with P?
ii) do all the vowels always occur together?
iii) do all the vowels never occur together?
iv) do the words begin with I and end in P?

Solution : There are 12 letters, of which 3 are Ns, 2 are Ds, 4 are Es, and the rest are all different. Therefore, the required number of arrangements is $\frac{12!}{3!2!4!} = 1663200$.

i) Let us fix P at the extreme left position. We, then, rearrange the remaining 11 letters consisting of 3 Ns, 2 Ds and 4 Es. So, the required number of words starting with $P = \frac{11!}{3!2!4!} = 138600$.

ii) *(Many teachers have told me that students find this part a problem. They usually need hints, a lot of questions and peer group discussion before all of them come to* *an understanding about it. So, let's see what this solution is.)* Since, all the vowels, 4 Es and 1 I, always have to occur together, we treat them as a single object. But, this single object can have several rearrangements within it, in fact, $\frac{5!}{4!} = 5$.

Now, this single object, together with the 7 remaining objects will account for 8 objects. A word will be formed if one arrangement of these 8 objects is taken, and following which one of the 5 arrangements of the vowels is taken. The 8 objects in which there are 3 Ns, 2 Ds , and the rest are different, can be rearranged in $\frac{8!}{3!2!}$ ways.

Therefore, by the multiplication principle, the required number of words is $= \frac{8!}{3!2!} \times \frac{5!}{4!} = 16800.$

Remark : Many learners again get confused here, giving the number as $\frac{8!}{3!2!} + \frac{5!}{4!}$. Here, you would need to give them several situations to help them realise that the task of getting one required arrangement is complete only when **corresponding to each of the choices** of $\frac{8!}{3!2!}$ arrangements, a choice of $\frac{5!}{4!}$ arrangements is made. This is why the multiplication principle is applied here.

Very often, a learner is not able to solve a problem because the language used is confusing or complicated.

iii) *(In this problem, the students get confused sometimes because of the language used. When we have asked them to explain their understanding, we find that they think the situation here is that* ***no two vowels*** *occur together. So, it needs to be explained to them that what is being asked for is that not all 5 vowels occur together. That is, 4 or 3 or 2 of them can occur together. Therefore, the arrangements to be considered here are all those that are not considered in Part (ii) above.)*

The required number is the total number of arrangements minus the total number of words in which all the vowels occur together
$= 1663200 - 16800 = 1646400.$

iv) Let us fix I at the extreme left position and P at the extreme right position. The required number of words is the number of arrangements of the remaining 10 letters in which there are 3 Ns, 2 Ds and 4 Es. This is $\frac{10!}{3!2!4!} = 12200$.

If your learners have reached this far with you, why don't you try the following exercises with them? Also analyse their solutions to evaluate how effective your teaching strategy has been.

E8) In how many ways can 2 blue balls, 3 red balls and 4 green balls be arranged in a row? All the balls are indistinguishable so far as their shape and size are concerned.

E9) How many distinct 6-digit numbers can be formed with the digits 1,2,1,2,0,2?

E10) In how many ways can the letters of the word PERMUTATIONS be arranged if the
- i) words start with P and end with S?
- ii) vowels are all together?
- iii) there are always 4 letters between P and S?

So far we have been discussing the arrangements of objects in a line, that is, counting linear permutations. Let us now see how students react to permutations of objects placed in a circle.

7.4.2 Circular Permutations

Suppose you ask your students, who are familiar with linear permutations, to count the number of distinct arrangements of 8 different objects positioned around a circle. Will their reply be 8!? Or, do they realise that many of these arrangements are really the same? Maybe, a good way to help them understand this is to divide the class into groups and to ask the groups to stand in circles. Then you could ask them if the arrangement can be considered different if each person in the circle moves to the position of the person to her left, without changing the order. You could contrast this with getting them to stand in rows, and the person from the front moving to the last position, which would give a different arrangement.

After doing a few such activities, you could ask your students questions like 'in how many ways can 6 beads, all of different colours, be threaded to form distinct necklaces? With many particular examples of this kind, they would gradually conclude that the number of circular permutations of n different objects is $\frac{n!}{n}$ i.e., (n-1)!.

You would also need to expose your learners to slightly more complicated problems like the following one. As always, you would need to allow them time and space to discuss their solutions with one another and with you.

Problem 7: In how many ways can 3 women and 4 men be seated at a round table if

i) all 3 women always sit together?
ii) all 3 women never sit together?
iii) no two women sit together?

Solution :

(i) Since, the three women have to sit together always, we can treat them as a single object. So in effect, we shall be left with 5 persons to be seated around a round table. This can be done in $(5-1)!$ ways. Now, corresponding to each of these arrangements, the three women can be rearranged in 3! ways. Therefore, using the multiplication principle, the required number of ways $= 4! \times 3! = 144$.

(ii) The total number of ways in which 3 women and 4 men could be seated around the table $= (4+3-1)! = 720$.
Also, we know that the number of ways in which all three women sit together $= 144$.
Therefore, the number of ways in which all three women never sit together $= 720 - 144 = 576$.

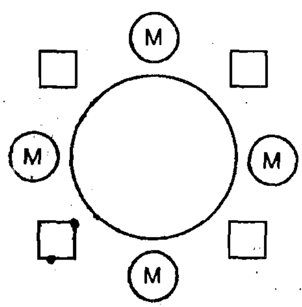

Fig. 4

(iii) First, the 4 men can be seated around the table in $(4-1)!$ ways. Corresponding to each of these arrangements, there are 4 places ( ) created (see Fig. 4), in 3 of which the three women are to be seated. This can be done in P(4,3) ways. Hence, by the fundamental principle of counting, the required number of ways $= 3! \times P(4,3) = 144$.

At this stage you may like to try the following exercises yourself, and with your learners.

E11) In how many ways can 3 men and 3 women be seated at a round table if

i) no restriction is imposed?
ii) each woman is to be between two men?
iii) two particular women must sit together?

E12) A tea party is arranged for 16 people along two sides of a long table with 8 chairs on each side. Four people wish to sit on a particular side and two on the other side. In how many ways can this be done?

E13) A box contains 5 red and 6 white balls. In how many ways can 6 balls be selected so that there are at least 2 balls of each colour?

Now let us took at why we can consider a permutation as a function. This aspect is missed out by most teachers, and therefore, by the students. But, mathematics is about 'making conceptions' and this is one connection that we can help our students make.

7.4.3 Permutation as a Function

Consider the set $A = \{R, O, S, E\}$ and any one-to-one function of A onto itself, say, $f : A \rightarrow A : f(R) = O, f(O) = S, f(S) = E$ and $f(E) = R$.

We see that under f the arrangement ROSE becomes OSER, a permutation of the objects of A.

Now take any permutation of the elements of A, say, SOER. Does this define a one-to-one function of A onto itself? It certainly does, namely,
$g : A \rightarrow A : g(R) = S, g(O) = O, g(S) = E, g(E) = R$.
You should check that there is a 1-to-1 correspondence between the bijective functions of A onto itself and the permutations of the elements of A.

Is this true for any finite set? This is what the following exercise is about.

E14) If there are n distinct elements in a set A, then show that every permutation on A is a one-to-one function of A onto itself. Hence show that the number of one-to-one functions of A onto itself is $n!$.

With this we come to the end of our discussion on communicating 'permutations' and 'combinations' to our students. Let us take a quick look at what we have discussed in this unit.

7.5 SUMMARY

In this unit, we have covered the following points.

1. We have discussed some points of confusion that students often face when dealing with problems that require the use of the multiplication principle. We have also given some examples of teaching strategies that may help children sort out these confusions.

2. We have suggested ways of helping children experience situations involving the counting of permutations and combinations, and how to help them understand what is required in problems based on these concepts.

3. We have discussed ways of explaining to your learners the idea of permutations of objects not necessarily distinct and ways of counting circular permutations.

4. We have explained how a permutation can be viewed as a one-to-one function.

7.6 COMMENTS ON EXERCISES

E1) This exercise is closely related to Unit 1. Also remember that most things can be understood by Class 6 children even, of course if they are not presented in a formal manner.

E2) Suppose we number the men as 1,2,3,…,n. Then the first man can be paired with any one of the n women, the second man can be paired with any one of the remaining (n−1) women, and so on. Hence, the number of ways of pairing is $n \times (n-1) \times .... \times 3 \times 2 \times 1 = n!$.

E3) For a number to be odd the last digit should be odd. So, the last position can be filled in 5 ways. Now, the leftmost position can be filled in 8 ways (digits from 1 to 9 except for the one in the units place). Then the middle position can be filled by a digit from 0 to 9 except the other two digits, that is, in 8 ways. Therefore, the required number = $8 \times 8 \times 5 = 320$.

E4) We will divide the required numbers into two sets. Set I consists of those numbers with the leftmost digit 6. Set II consists of those numbers that have the leftmost digit greater than 6. In Set I the number of elements is 1.4.6.5.4 (the 1st digit is chosen in 1 way, the 2nd can be only from 5,7,8,9, the third in 6 ways, etc.). The number of elements in Set I is, thus, 480.

In Set II we have 3.7.6.5.4 = 2520 numbers.

Thus the required answer is 480 + 2520 = 3000.

E5) For answering this, we expect you to talk to students individually or in small groups in a friendly environment. The aim would be to find out their understanding behind their solutions to the problems. Only then can you think of ways of correcting this understanding.

E6) Would you be applying the principles of moving from concrete to abstract, and from particular to general? What kind of inductive and deductive reasoning would you use that the students would understand, and be convinced by?

E7) Consider all the permutations of n objects, where p_1 objects are of Type-I, p_2 objects are of Type-II, p_k objects are of Type-k, and the remaining $n - (p_1+p_2+\ldots+p_k)$ are all distinct or $n = p_1+p_2+\ldots+ p_k$. The number of permutations of this collection of objects is $\dfrac{n!}{p_1!p_2!p_3!...p_k!}$.

E8) Explain why the required number of ways = $\dfrac{9!}{2!3!4!}$.

E9) There are two 1s, three 2s and one 0. They can be rearranged in $\frac{6!}{2!3!} = 60$ ways. But this includes those numbers which have 0 at the extreme left position, which are $= \frac{5!}{2!3!} = 10$ in number.

Hence, the required number is $= 60-10 = 50$.

E10) i) As the positions of P and S are fixed, the 10 remaining letters, in which there are two Ts and the rest are different, can be arranged in $\frac{10!}{2!}$ ways.

ii) Let us consider the five vowels E,U,A,I,O as one object. Then we have 8 things to be arranged in which there are 2 Ts and the rest are different. These can be arranged in $\frac{8!}{2!}$ ways.

Also, the 5 vowels can be arranged among themselves in 5! ways.

Hence, the required number of arrangements $= \frac{8!}{2!} \times 5!$.

iii) The positions of P and S could be as follows:

P ---- S ------
- P ---- S -----
--P ---- S ----
--- P ----S ---
----P ---- S--
-----P ---- S -
------P ----S

We also need to include all such cases in which the positions of P and S are interchanged.

Therefore, there are $7 \times 2 = 14$ such ways. In each of these arrangements, the remaining 10 letters, having 2 Ts can be arranged in $\frac{10!}{2!}$ ways.

Therefore, the required number of arrangements $= 14 \times \frac{10!}{2!}$.

E11) i) Six persons can be seated round a table in $5! = 120$ ways.

ii) First, three men can be seated in 2! ways. Then, the three women can be seated in three places (in Fig. 5) created in 3! ways. Therefore, the required number of ways $= 2! \times 3! = 12$.

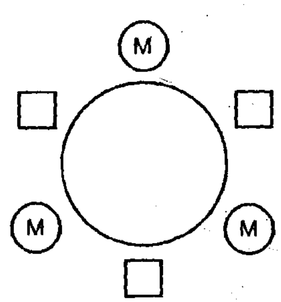

Fig. 5

iii) Temporarily, let us consider two particular women as one woman. Then we have 5 persons to be seated round a table, which can be done in $4! = 24$ ways. The two women can be arranged among themselves in 2! ways. Therefore, the required number of ways $= 24 \times 2 = 48$.

E12) Let the 4 fixed people be seated on one side, say Side I. This can be in P(8,4) ways. The two people to be seated on the other side, say Side II, can be seated in P(8,2) ways. Therefore, the number of ways these 6 people can be seated $= P(8,4) \times P(8,2)$.

Corresponding to each of these arrangements, the rest of the 10 seats will be occupied by the remaining 10 people in 10! ways.

Therefore, the required number of ways = $P(8,4) \times P(8,2) \times 10!$.

E13) The following categories of combinations are possible:

i) 2 red and 4 white balls:
This can be done in $C(5,2) \times C(6,4)$ ways.

ii) 3 red and 3 white balls:
This can be done in $C(5,3) \times C(6,3)$ ways.

iii) 4 red and 2 white balls:
This can be done in $C(5,4) \times C(6,2)$ ways.

Therefore, the required number of ways = $(C(5,2) \times C(6,4)) + (C(5,3) \times C(6,3)) + (C(5,4) \times C(6,2))$.

An alternative route would be :

Total number of ways of drawing 6 balls = ${}^{11}C_6$

From this we subtract the cases we do not require, namely, drawing 1 or 0 balls of each colour.

So, the required number is

$${}^{11}C_6 - [(C(5,1) \times C(6,5) + (C(5,5) \times C(6,1) + (C(5,0) \times C(6,6))] = 425.$$

E14) This can be done, as we've shown above the exercise for n=4. Can you apply the principle of induction here?

Since you know the number of permutations of n objects is n!, the second statement follows.